FIRST PRACTICE EXAM SECOND HOURLY **Math 21b, Fall 2003**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T F Similar matrices have the same determinant.

Solution:
$\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \det(A)$.

T F The matrix $\begin{bmatrix} 1 & 100 & 1 & 1 \\ 100 & 1 & 1 & 1 \\ 1 & 1 & 1 & 100 \\ 1 & 1 & 100 & 1 \end{bmatrix}$ is invertible.

Solution:
One pattern dominates clearly.

T F If A is a 3×3 matrix for which every entry is 1, then $\det(A) = 1$.

Solution:
The kernel is nontrivial, contains for example $[1, -1, 0]^T$.

T F If $\vec{v}$ is an eigenvector of A and of B and A is invertible, then $\vec{v}$ is an eigenvector of $3A^{-1} + 2B$.

Solution:
$\vec{v}$ is also an eigenvector of A^{-1}.

T F $\det(-A) = \det(A)$ for every 5×5 matrix A.

Solution:
For odd n, an $n \times n$ matrix satisfies $\det(-A) = -\det(A)$.

T F If $\vec{v}$ is an eigenvector of A and an eigenvector of B and A is invertible, then $\vec{v}$ is an eigenvector of $A^{-3}B^2$.

Solution:
Again, one can add and multiply matrices with the same eigenvector and get matrices with the same eigenvector.

T F If a 11×11 matrix has the eigenvalues $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, then A is diagonalizable.

Solution:
All eigenvalues are different.

T | **F**

For any $n \times n$ matrix, the matrix A has the same eigenvectors as A^T.

Solution:
The same eigenvalues yes, but not the same eigenvectors.

T | F

If $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector of length 1, then $\vec{v}\vec{v}^T$ is a diagonalizable 3×3 matrix.

Solution:
The matrix is a projection matrix onto a one dimensional line. In suitable coordinates, this matrix has only 1 or 0 in the diagonal.

T | F

The span of m orthonormal vectors is m-dimensional.

Solution:
Yes, orthogonal vectors can not be linearly dependent.

T | F

A square matrix A can always be expressed as the sum of a symmetric matrix and a skew-symmetric matrix as follows $A = \frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$.

Solution:
Indeed, $A + A^T$ is symmetric and $A - A^T$ is skew symmetric and the sum as given holds.

T | **F**

There exists an invertible $n \times n$ matrix A which satisfies $A^TA = A^2$, but A is not symmetric.

Solution:
Because A is invertible, one can multiply the equation from the right with A^{-1} and gets $A^T = A$.

T | F

If two $n \times n$ matrices A and B commute, then $(A^T)^2$ commutes with $(B^3)^T$.

Solution:
If A and B commute, then A^T commutes with B^T and therefore $(A^n)^T$ commutes with $(B^m)^T$ for all m, n. Note that A^T does not necessarily commute with B.

T | **F**

$-AA^T$ is skew-symmetric for every $n \times n$ matrix A.

Solution:
$B = -AA^T$ satisfies $B^T = -AA^T = B$ so $-AA^T$ is symmetric.

T | F

A matrix A maps a least squares solution $\vec{x}^*$ of $A\vec{x} = \vec{b}$ to the projection of $\vec{b}$ onto $\text{im}(A)$.

Solution:
This was the basic insight behind the least square story.

T | F

A matrix which is obtained from the identity matrix by an arbitrary number of switching of rows or columns is an orthogonal matrix.

Solution:
Indeed, switching of columns or rows does not change the orthogonality of each pair of vectors.

T | F

The trace of a real skew-symmetric $n \times n$ matrix is always equal to 0.

Solution:
Indeed, the diagonal has to be zero.

T | **F**

There exists a real 3×3 matrix A which satisfies $A^4 = -I_3$.

Solution:
Take determinants to see that this is not possible.

T | F

If A has four different integer eigenvalues, then there exists a vector $\vec{v}$ such that $||A^n\vec{v}|| \to \infty$ for $n \to \infty$.

Solution:
Then one eigenvalue has to be different from $0, 1, -1$ and its absolute value be bigger than 1. Now, with its eigenvector $\vec{v}$ one has $A^n\vec{v} = \lambda^n\vec{v}$ which becomes larger and larger in absolute value.

T | **F**

Given 5 data points $(x_1, y_1), ..., (x_5, y_5)$, then a best fit with a polynomial $a + bt + ct^2 + dt^3 + et^4 + ft^5$ is possible in a unique way.

Solution:
There are 6 variables and 5 conditions to be satisfied, one has in this case many possibilities to fit the data exactly without error.

Problem 3) (10 points)

Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$.

a) Find all eigenvalues of A with their multiplicities.

b) Write down all eigenvectors of A. What are the geometric multiplicities of the eigenvalues?

c) What is $\det(A)$.

Hint. You might want to look at $A + I_5$ first.

Solution:
a) $A + I_5$ has the eigenvalue 5 with multiplicity 1 and the eigenvalue 0 with multiplicity 4. So A has the eigenvalue 4 with multiplicity 1 and the eigenvalue -1 with multiplicity 4.
b) The eigenvectors of $A + 1$ are the same as the eigenvectors of A. The eigenvector to 4 is $[1,1,1,1,1]^T$. The eigenvectors to -1 are $[1,-1,0,0,0], [1,0,-1,0,0], [1,0,0,-1,0], [1,0,0,0,-1]$.
c) Take the product of the eigenvalues, which is 4.

Problem 4) (10 points)

a) Let A be a $n \times n$ matrix such that $A^2 = 2A - I$. What are the possible eigenvalues of A?

b) Let A be a real $n \times n$ matrix such that $A^4 = -I_n$. Show that n must be even.

Solution:
a) λ has to be a root of the polynomial $\lambda^2 = 2\lambda - 1$. This means 1 is the only possible eigenvalue.
b) Look at the determinant of A. If n is odd, the determinant of the left hand side is nonnegative, on the right hand side it is negative.

Problem 5) (10 points)

Let $A_n = \begin{bmatrix} 1 & 1 \\ 0 & 1+1/n \end{bmatrix}$, where n is a positive integer.

a) Find all eigenvalues for A_n.

b) Find an eigenbasis A_n for each n.

c) We hope to get an eigenbasis for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ by taking the limit in b), when n goes to infinity and possibly rescaling the eigenvectors so that they converge. Does this work? What happens?

Solution:
a) The eigenvalues are $1, 1 + 1/n$.
b) The eigenvectors are $[1,0]^T$ and $[n,1]^T$.
c) The second eigenvalue converges to the first one. The second eigenvector diverges, but rescaling it gives eigenvectors $[1, 1/n]$ which converges to the first eigenvector. In the limit, we have only one eigenvector. As n increases, the two eigenvectors become more alligned.

Problem 6) (10 points)

Given $n+1$ numbers $a_0, a_1, ..., a_n$, define the matrix $A = \begin{bmatrix} 1 & 1 & ... & 1 & 1 \\ a_0 & a_1 & ... & a_{n-1} & a_n \\ a_0^2 & a_1^2 & ... & a_{n-1}^2 & a_n^2 \\ ... & ... & ... & ... & ... \\ a_0^n & a_1^n & ... & a_{n-1}^n & a_n^n \end{bmatrix}$. It is called a **van der Monde matrix**.

a) If we call $a_n = x$, then the determinant of A becomes a function of x. Show that is a polynomial of degree n in x.

b) Verify that $f(a_0) = f(a_1) = ... = f(a_{n-1}) = 0$.

c) Conclude that $f(x) = k(x - a_1) \ldots, (x - a_{n-1})$ for some constant k.

d) Verify that the constant k is a determinant of a $n \times n$ van der Monde matrix.

e) Use this to conclude that $\det(A) = \prod_{i>j}(a_i - a_j)$.

Solution:
a) The degree can not be larger than n and since a_n^n occurs in the determinant, the degree is n.
b) If a_n is equal to any other a_i, then we have two identical columns and so zero determinant.
c) By the fundamental theorem of algebra, we can factor out the roots and there are exactly n roots.
d) Yes, it is the determinant of the corresponding van der Monde matrix with $a_0, ..., a_{n-1}$. To get the constant k, note that this constant is the coefficient in front of the x^n in the polynomial and this is by the Laplace expansion with respect to the last row the determinant of the matrix, where the last row and last column are cut away.
e) Use induction in n starting with $n = 0$, where the determinant is 1.
To see the argument better, consider the case

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a_0 & a_1 & a_2 \\ a_0^2 & a_1^2 & a_2^2 \end{bmatrix} .$$

The first step was to replace a_2 with the variable x and compute the determinant

$$A(x) = \begin{bmatrix} 1 & 1 & 1 \\ a_0 & a_1 & x \\ a_0^2 & a_1^2 & x^2 \end{bmatrix}$$

which is a polynomial of degree 2 in x. It can be written as $(a_1 - a_0)(x - a_0)(x - a_1) = k(x - a_0)(x - a_1)$ and $k = (a_1 - a_0)$ is the determinant of the matrix

$$A(x) = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix} .$$

Problem 7) (10 points)

A discrete dynamical system is given by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - 2y \\ -x + 3y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} .$$

Find a closed formula for $T^{100}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$.

Solution:
The eigenvector to the eigenvalue 4 is $\vec{v} = [-1, 1]^T$. The eigenvector to the eigenvalue 1 is $\vec{w} = [2, 1]^T$. Because $[1, 1]^T = \vec{v} + \vec{w}$, we have $T^{100}(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = 4^{100} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1^{100} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Problem 8) (10 points)

a) Show that for an arbitrary matrix A for which $A^T A$ is invertible, the least squares solution of $A\vec{x} = \vec{b}$ simplifies to

$$\vec{x} = R^{-1} Q^T \vec{b} ,$$

if $A = QR$ is the QR decomposition of A.

b) Find the least square solution in the case $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ using this formula.

Solution:
a) This is a direct algebraic computation, using $Q^T Q = I$:

$$(A^T A)^{-1} A^T = (R^T Q^T Q R)^{-1} R^T Q^T = R^{-1} (R^T)^{-1} R^T Q^T = R^{-1} Q^T .$$

b) The QR decomposition is $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2^{-1/2} \\ 0 & 2^{-1/2} \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$.

Problem 9) (10 points)

Find the least square solution for the system $A\vec{x} = \vec{b}$ given by the equations

$$\begin{aligned} x + y &= 4 \\ y &= 2 \\ x &= -1 . \end{aligned}$$

Solution:
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$. Form $\vec{x} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.