11/19/2003 SECOND HOURLY **Math 21b, Fall 2003**

Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T F If x^* is the least squares solution of $Ax = b$, then $||b||^2 = ||Ax^*||^2 + ||b - Ax^*||^2$.

Solution:
This is Pythagoras applied to the orthogonal vectors $b - Ax^*$, Ax^*.

T F If c is an eigenvalue of A, then c^3 is an eigenvalue of A^3.

Solution:
$A\vec{v} = c\vec{v}$, then $A^3\vec{v} = c^3\vec{v}$.

T F A shear in the plane is not diagonalizable.

Solution:
Indeed, the shear in the plane is the prototype of a nondiagonalizable transformation.

T F There exists an invertible 5×5 matrix A for which $SAS^{-1} = -A$ for some S.

Solution:
Looking at the determinant shows $\det(S^{-1}AS) = \det(A) = \det(-A)$ which is only possible for $n \times n$ matrices with even n.

T F If A is a 3×3 matrix, then $\det(3A) = 3\det(A)$.

Solution:
We have $\det(3A) = 3^3\det(A)$.

T F If A is an invertible $n \times n$ matrix, then A is diagonalizable.

Solution:
The shear is invertible but not diagonalizable.

T F A projection is an orthogonal transformation.

Solution:
Any orthogonal transformation is invertible.

T **F** The determinant of an orthogonal matrix always equals to 1.

Solution:
It can also be equal to -1.

T **F** Any two matrices whose eigenvalues are equal must be similar.

Solution:
The identity matrix and the shear have the eigenvalues 1 but they are not similar.

T **F** Any two matrices whose eigenvectors are equal must be similar.

Solution:
Two different diagonal matrices have the same eigenvectors but are not similar

T F Let R be a 3×3 rotation matrix. Then $\det(R - I_3) = 0$.

Solution:
Because 1 is an eigenvalue, $\det(R - I_3) = 0$.

T F The standard basis vectors of $\mathbf{R}^n$ are the eigenvectors of every diagonal matrix.

Solution:
Obvious, in the k'th row, we have a multiple of the k'th basis vector.

T F $\ker(A^T) = \ker(AA^T)$ for every matrix A.

Solution:
You have proven this in a homework.

T F For every 2×2 matrix with determinant 2, one knows that $|A^n \vec{v}|$ grows to infinity for at least one vector $\vec{v}$.

Solution:
Because the product of the eigenvalues is the determinant, there is an eigenvalue which has absolute value > 1. Take the eigenvector $\vec{v}$ of that eigenvalue.

T F For any $n \times n$ matrix, the matrix A has the same eigenvalues as A^T.

Solution:
They have the same characteristic polynomial

T F If $A^2 = A$, then every eigenvalue λ of A is either $\lambda = 1$ or $\lambda = 0$.

Solution:
If A has an eigenvalue λ, then for the corresponding eigenvector $\vec{v}$ we have $A^2\vec{v} = \lambda^2\vec{v}$ and $A\vec{v} = \lambda\vec{v}$. So, $\lambda^2 = \lambda$.

T F An upper triangular matrix is invertible if and only if all of its diagonal entries are non-zero

Solution:
The determinant is the product of the diagonal elements.

T F All projection matrices are diagonalizable.

Solution:
One can chose a basis, in which part of the basis is an orthogonal set in V and part of the basis is an orthogonal set in $V^\perp$. In that basis, the matrix is diagonal.

T F If A is invertible, then A and A^{-1} have the same eigenvectors.

Solution:
Yes, if $A\vec{v} = \lambda\vec{v}$, then $A^{-1}\vec{v} = \lambda^{-1}\vec{v}$.

T F If the characteristic polynomial of a 3×3 matrix A is $(\lambda - 1)^2(\lambda - 2)$, then A is similar to either $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution:
This is a consequence of the Jordan normal form theorem, treated in class (not in the book nor in handouts).

	Total

Problem 2) (10 points)

Check the boxes of all matrices which have zero determinants. You don't have to give justifications.

a) ☐ $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ b) ☐ $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

c) ☐ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ d) ☐ $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

e) ☐ $A = \begin{bmatrix} 10^{100} & 1 & 1 & 1 \\ 1 & 10^{100} & 1 & 1 \\ 1 & 1 & 10^{100} & 0 \\ 1 & 0 & 1 & 10^{100} \end{bmatrix}$ f) ☐ $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

g) ☐ $A = \begin{bmatrix} 11 & 10 & 8 & 5 \\ 9 & 7 & 4 & 0 \\ 6 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$ h) ☐ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

i) ☐ $A = \begin{bmatrix} 1 & 1 & 7 & 0 \\ 1 & 6 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ j) ☐ $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \end{bmatrix}$

Solution:
b),c),f),j) have zero determinant. Some hints:
a) One pattern is nonzero only.
b) The differences between consecutive rows is the same vector.
c) Two rows are identical.
d) One pattern is nonzero only.
e) One pattern dominates clearly.
f) Partitioned matrix, the matrix at the lower right corner is noninvertible.
g) One pattern, the anti-diagonal is nonzero only.
h) You have constructed this matrix as the one which maximizes the determinant among all matrices with entries $0, -1, 1$.
i) One pattern only. Can also be seen as a partitioned matrix.
j) Parallel columns.

Problem 3) (10 points)

a) Find all (possibly complex) eigenvalues and eigenvectors of the matrix $Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) Verify that Q^T has the same eigenvectors as Q.

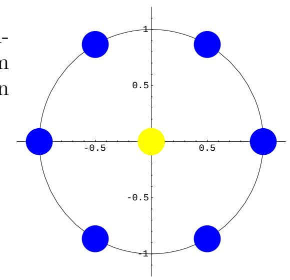
c) Find a diagonal matrix B which is similar to the symmetric matrix

$$A = Q + Q^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} .$$

State algebraic and geometric multiplicities of the eigenvalues.

Hint. You have similar example as in a) in class and in a practice exam.

Solution:



a) The characteristic polynomial is $\lambda^6 - 1$. The eigenvalues are the $6'th$ roots of 1 and are of the form $\lambda_k = e^{i2\pi k/6}$. For each of these eigenvalues λ, there is an eigenvector $\vec{v}_\lambda = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^2 \\ \lambda^4 \\ \lambda^5 \end{bmatrix}$.

($Q\vec{v} = \begin{bmatrix} a_5 \\ 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \lambda\vec{v}$ implies $a_1 = \lambda, a_2 = \lambda a_1, a_3 = \lambda a_2, a_4 = \lambda a_3, a_5 = \lambda a_4$.

b) Because $Q^T = Q^{-1}$, both Q and Q^T have the same eigenvector: if $Q\vec{v} = \lambda\vec{v}$, then $Q^T\vec{v} = Q^{-1}\vec{v} = \lambda^{-1}\vec{v}$.
So, if λ is an eigenvalue with eigenvector $\vec{v}$, then $\lambda + \lambda^{-1}$ is an eigenvalue of A with eigenvector $\vec{v}$.
c) The matrix A has the eigenvalue $2\cos(2\pi k/6)$. These are $-2, 2$ with algebraic multiplicity 1 and $-1, 1$ with algebraic multiplicity 2.

The matrix is similar to $B = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Remark (but this was not asked).
While the complex eigenvectors $\vec{v}_\lambda$ are also eigenvectors of A, there are also **real** eigenvectors of A. $\lambda_1 = -2, \vec{v} = [-1, 1, -1, 1, -1, 1]$, $\lambda_2 = 2, \vec{v} = [1, 1, 1, 1, 1, 1]$, $\lambda_3 = -1, \vec{v} = [-1, 0, 1, -1, 0, 1]$ and $\vec{v} = [0, 1, -1, 0, 1, -1]$ $\lambda_4 = 1, \vec{v} = [1, 0, -1, -1, 0, 1]$ and $\vec{v} = [0, -1, -1, 0, 1, 1]$.

Problem 4) (10 points)

Find the function $f(t) = a + bt$ which best fits the data

$$\begin{aligned} (x_1, y_1) &= (-1, 1) \\ (x_2, y_2) &= (0, 2) \\ (x_3, y_3) &= (1, 2) \\ (x_4, y_4) &= (3, 1) \\ (x_5, y_5) &= (3, 0) \end{aligned}$$

Solution:
Setting up the equations $a + bx_i = y_i$ gives the system

$$\begin{aligned} a + -b &= 1 \\ a &= 2 \\ a + b &= 2 \\ a + 3b &= 1 \\ a + 3b &= 0 \end{aligned}$$

which can be written as $A\vec{x} = \vec{b}$, where $\vec{x} = [a, b]^T$ and $\vec{b} = [1, 2, 2, 1, 0]^T$. The least square solution is given by $(A^TA)^{-1}A^T\vec{b}$. One has $A^TA = \begin{bmatrix} 5 & 6 \\ 6 & 20 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 3/2 \\ -1/4 \end{bmatrix}$. The best fit is given by the function $\boxed{f(t) = 3/2 - t/4}$.

Problem 5) (10 points)

Find a basis for the subspace V of $\mathbf{R}^4$ given by the equation $x + 2y + 3z + 4w = 0$. Find the matrix which gives the orthogonal projection onto this subspace.

Hint: The problem can be done in different ways. Choosing an orthonormal basis in V simplifies some computations.

Solution:
1. Solution.
Form $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and define $P' = A(A^TA)^{-1}A^T$ which is the projection onto the orthogonal complement. We get $(A^TA)^{-1} = 1/30$ and $P' = AA^T/30 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}/30$ so that $P = 1 - P'$ and $P = \begin{bmatrix} 29 & -2 & -3 & -4 \\ -2 & 26 & -6 & -8 \\ -3 & -6 & 21 & -12 \\ -4 & -8 & -12 & 14 \end{bmatrix}/30$.
2. Solution.
Find directly an orthogonal basis in V, for example by choosing two already orthogonal vectors $[-2, 1, 0, 0], [0, 0, -4, 3]$ and searching for a third $[b, 2b, 3a, 4a]$ which is orthogonal to the first two vectors and in V if $b + 4b + 9a + 16a = 0$, which gives $25a = -5b$ or $5a = b$ and so $[5, 10, -3, -4]$. These vectors can be made orthonormal $[-2, 1, 0, 0]/\sqrt{5}, [0, 0, -4, 3]/5, [5, 10, -3, -4]/(5\sqrt{6})$. We can now form the matrix $A = \begin{bmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & \sqrt{2/3} \\ 0 & -4/5 & -\sqrt{3/2}/5 \\ 0 & 3/5 & -2\sqrt{2/3}/5 \end{bmatrix}$ and then AA^T and obtain $P = AA^T$.
3. Solution.
Take three arbitrary linearly independent vectors in V, form the matrix A which contains these vectors as columns and then calculate $A(A^TA)^{-1}A^T$. While routine, this gives quite a bit of work, more than in the 1. and 2. Solutions.

Problem 6) (10 points)

a) (7 points) Find the QR decomposition of

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

then form the new matrix $T(A) = RQ$.

b) (3 points) Verify that for any invertible $n \times n$ matrix $A = QR$, the matrix $T(A) = RQ$ has the same eigenvalues as A.

Solution:
a) $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. $R = \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$. $RQ = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$.
b) Take $S = Q$, then $ST(A)S^{-1} = QRQQ^{-1} = QR$, so that $T(A) = RQ$ is similar to $A = QR$ and has the same spectrum as A.

Problem 7) (10 points)

Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & 0 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & 0 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & 0 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & 0 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & 0 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{bmatrix}.$$

Show your work carefully.

Solution:
First solution.
Adding the first row to the others gives a upper triangular matrix with entries $1, 2, 3, 4, 5, 6, 7, 8$ in the diagonal. The answer is $8! = 40'320$.
Second solution.
Look at the matrix as a partitioned matrix, which is made of 2×2 matrices. Notice, that most of these 2×2 matrices have zero determinant. Each of these can be replaced by the zero matrix without changing the determinant. We end up with a matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 \end{bmatrix}$$

which has determinant 8! too.
Third solution.
Look at the matrix as a partitioned matrix made up of 4 matrices of size 4×4. The anti diagonal matrices have zero determinant and can be replaced by the zero matrix without changing the determinant.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 4 & 0 & 0 & 0 & 0 \\ -1 & -2 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & -5 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & -5 & -6 & 0 & 8 \\ 0 & 0 & 0 & 0 & -5 & -6 & -7 & 0 \end{bmatrix}$$

One can apply now the same trick to the two 4×4 matrices, split it up into 2.

Problem 8) (10 points)

The growth of the rose bush is modeled by a vector $\begin{bmatrix} n(t) \\ a(t) \end{bmatrix}$ which gives the number of new and old branches. The time evolution was described by a matrix A such that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find a closed formula for $n(t)$ and $a(t)$ if $n(0) = 1$ and $a(0) = 0$.

Hint. You have solved this problem in a homework, but the lila bush is now a rose bush.

Solution:
The matrix is $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. The eigenvector to the eigenvalue 2 is $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the eigenvector to the eigenvalue -1 is $\vec{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Because $(\vec{v} - \vec{w})/3$ is the initial condition, we know $A^n[1, 0]^T = A^n\vec{v}A^n\vec{w} = 2^n\vec{v}/3 - (-1)^n\vec{w}/3$.

Problem 9) (10 points)

Construct an orthonormal basis for the three-dimensional subspace of $\mathbf{R}^4$ spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and call the vectors of your basis $\vec{w}_1$,$\vec{w}_2$ and $\vec{w}_3$.

Let $A = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix}$. Compute $\det(AA^T)$ and $\det(A^TA)$ without multiplying out these matrices.

Solution:
a) AA^T is a 4×4 matrix which has its image contained in the image of A. It is the projection from a four dimensional space to a three dimensional space and therefore not invertible. Therefore $\boxed{\det(AA^T) = 0}$.
b) A^TA is the 3×3 identity matrix and has determinant $\boxed{\det(A^TA) = 1}$.