Name:

MWF10 Izzet Coskun

MWF11 Oliver Knill

- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

| T | F | Similar matrices have the same determinant. |

| T | F | The matrix $\begin{bmatrix} 1 & 100 & 1 & 1 \\ 100 & 1 & 1 & 1 \\ 1 & 1 & 1 & 100 \\ 1 & 1 & 100 & 1 \end{bmatrix}$ is invertible. |

| T | F | If A is a 3×3 matrix for which every entry is 1, then $\det(A) = 1$. |

| T | F | If $\vec{v}$ is an eigenvector of A and of B and A is invertible, then $\vec{v}$ is an eigenvector of $3A^{-1} + 2B$. |

| T | F | $\det(-A) = \det(A)$ for every 5×5 matrix A. |

| T | F | If $\vec{v}$ is an eigenvector of A and an eigenvector of B and A is invertible, then $\vec{v}$ is an eigenvector of $A^{-3}B^2$. |

| T | F | If a 11×11 matrix has the eigenvalues $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$, then A is diagonalizable. |

| T | F | For any $n \times n$ matrix, the matrix A has the same eigenvectors as A^T. |

| T | F | If $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector of length 1, then $\vec{v}\vec{v}^T$ is a diagonalizable 3×3 matrix. |

| T | F | The span of m orthonormal vectors is m-dimensional. |

| T | F | A square matrix A can always be expressed as the sum of a symmetric matrix and a skew-symmetric matrix as follows $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$. |

| T | F | There exists an invertible $n \times n$ matrix A which satisfies $A^T A = A^2$, but A is not symmetric. |

| T | F | If two $n \times n$ matrices A and B commute, then $(A^T)^2$ commutes with $(B^3)^T$. |

| T | F | $-AA^T$ is skew-symmetric for every $n \times n$ matrix A. |

| T | F | A maps a least squares solution $\vec{x}^*$ of $A\vec{x} = \vec{b}$ to the projection of $\vec{b}$ onto $\text{im}(A)$. |

| T | F | A matrix which is obtained from the identity matrix by an arbitrary number of switching of rows or columns is an orthogonal matrix. |

| T | F | The trace of a real skew-symmetric $n \times n$ matrix is always equal to 0. |

| T | F | There exists a real 3×3 matrix A which satisfies $A^4 = -I_3$. |

| T | F | If A has four different integer eigenvalues, then there exists a vector $\vec{v}$ such that $||A^n\vec{v}|| \to \infty$ for $n \to \infty$. |

| T | F | Given 5 data points $(x_1, y_1), ..., (x_5, y_5)$, then a best fit with a polynomial $a + bt + ct^2 + dt^3 + et^4 + ft^5$ is possible in a unique way. |

| | Total |

Problem 3) (10 points)

Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$.

a) Find all eigenvalues of A with their multiplicities.

b) Write down all eigenvectors of A. What are the geometric multiplicities of the eigenvalues?

c) What is $\det(A)$.

Hint. You might want to look at $A + I_5$ first.

Problem 4) (10 points)

a) Let A be a $n \times n$ matrix such that $A^2 = 2A - I$. What are the possible eigenvalues of A?

b) Let A be a real $n \times n$ matrix such that $A^4 = -I_n$. Show that n must be even.

Problem 5) (10 points)

Let $A_n = \begin{bmatrix} 1 & 1 \\ 0 & 1 + 1/n \end{bmatrix}$, where n is a positive integer.

a) Find all eigenvalues for A_n.

b) Find an eigenbasis A_n for each n.

c) We hope to get an eigenbasis for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ by taking the limit in b), when n goes to infinity and possibly rescaling the eigenvectors so that they converge. Does this work? What happens?

Problem 6) (10 points)

Given $n+1$ numbers $a_0, a_1, ..., a_n$, define the matrix $A = \begin{bmatrix} 1 & 1 & ... & 1 & 1 \\ a_0 & a_1 & ... & a_{n-1} & a_n \\ a_0^2 & a_1^2 & ... & a_{n-1}^2 & a_n^2 \\ ... & ... & ... & ... & ... \\ a_0^n & a_1^n & ... & a_{n-1}^n & a_n^n \end{bmatrix}$. It is called

a **van der Monde matrix**.

a) If we call $a_n = x$, then the determinant of A becomes a function of x. Show that is a polynomial of degree n in x.

b) Verify that $f(a_0) = f(a_1) = ... = f(a_{n-1}) = 0$.

c) Conclude that $f(x) = k(x - a_1) ..., (x - a_{n-1})$ for some constant k.

d) Verify that the constant k is a determinant of a $n \times n$ van der Monde matrix.

e) Use this to conclude that $\det(A) = \prod_{i>j}(a_i - a_j)$.

Problem 7) (10 points)

A discrete dynamical system is given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 2y \\ -x + 3y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}.$$

Find a closed formula for $T^{100}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$.

Problem 8) (10 points)

a) Show that for an arbitrary matrix A for which $A^T A$ is invertible, the least squares solution of $A\vec{x} = \vec{b}$ simplifies to

$$\vec{x} = R^{-1}Q^T\vec{b},$$

if $A = QR$ is the QR decomposition of A.

b) Find the least square solution in the case $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ using this formula.

Problem 9) (10 points)

Find the least square solution for the system $A\vec{x} = \vec{b}$ given by the equations

$$\begin{aligned} x + y &= 4 \\ y &= 2 \\ x &= -1 . \end{aligned}$$