10/22/2003 FIRST HOURLY PRACTICE **Math 21b, Fall 2003**

Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. (The actual exam will have more free space). If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T	**F**	Suppose A is an $m \times n$ matrix, where $n < m$. If the rank of A is m, then there is a vector $y \in \mathbf{R}^m$ for which the system $Ax = y$ has no solutions.
T	**F**	The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ is invertible.
T	**F**	The rank of an lower-triangular matrix equals the number of non-zero entries along the diagonal.
T	**F**	The row reduced echelon form of a 3×3 matrix of rank 2 is one of the following $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
T	**F**	The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a shear.
T	F	For any matrix A, one has $\dim(\ker(A)) = \dim(\ker(\mathrm{rref}(A)))$.
T	**F**	If $\ker(A)$ is included in $\mathrm{im}(A)$, then A is not invertible.
T	F	There exists an invertible 3×3 matrix, for which 7 of the 9 entries are π.
T	F	The dimension of the image of a matrix A is equal to the dimension of the image of the matrix $\mathrm{rref}(A)$.
T	**F**	There exists an invertible $n \times n$ matrix whose inverse has rank $n - 1$.
T	F	If A and B are $n \times n$ matrices, then AB is invertible if and only if both A and B are invertible.
T	**F**	There exist matrices A, B such that A has rank 4 and B has rank 7 and AB has rank 5.
T	F	There exist matrices A, B such that A has rank 2 and B has rank 7 and AB has rank 1.
T	F	If for an invertible matrix A one has $A^2 = A$, then $A = I_2$.
T	**F**	If an invertible matrix A satisfies $A^2 = 1$, then $A = I_2$ or $A = -I_2$.
T	F	The matrix $\begin{bmatrix} c-1 & -1 \\ 2 & c+1 \end{bmatrix}$ is invertible for every real number c.
T	**F**	For 2×2 matrices A and B, if $AB = 0$, then either $A = 0$ or $B = 0$.
T	F	The determinant of a shear in the plane is always 1.
T	**F**	The plane $x + y - z = 1$ is a linear subspace of three dimensional space.
T	**F**	If T is a rotation in space around an angle $\pi/6$ around the z axes, then the linear transformation $S(x) = T(x) - x$ is invertible.

Problem 2) (10 points)

Determine for each of the following matrices A, whether the system $A\vec{x} = \vec{e}_1$ has zero, one or infinitely many solutions and find the dimension of the image of A in each case:

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

c) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

d) $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

e) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

f) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

g) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution:
a) Infinite. $\dim(\mathrm{im}(A)) = 2$.
b) One. $\dim(\mathrm{im}(A)) = 3$.
c) Infinite. $\dim(\mathrm{im}(A)) = 2$
d) One. $\dim(\mathrm{im}(A)) = 3$.
e) Infinite. $\dim(\mathrm{im}(A)) = 2$.
f) Infinite. $\dim(\mathrm{im}(A)) = 1$.
g) One. $\dim(\mathrm{im}(A)) = 3$.

Problem 3) (10 points)

a) Write the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as a product of a rotation and a dilation.

b) What is the length of the vector $\vec{v} = A^{100}e_1$, where e_1 is the first basis vector?

c) In which direction does the vector $\vec{v}$ point?

d) Find a matrix B such that $B^2 = A$.

Solution:
a) The matrix is a rotation dilation matrix, a rotation by $\pi/4$ and scaling by $\sqrt{2}$.
b) A^{100} is a composition of a scaling by a factor $\sqrt{2}^{100} = 2^{50}$ and rotation by π. So, $A^{100} = \begin{bmatrix} -2^{50} & 0 \\ 0 & -2^{50} \end{bmatrix}$.
c) It points to $-e_1$: after each 8 rotations, we are back to the initial position, so also after 96 rotations. The additional 4 rotations turn the vector to $-e_1$
d) A rotation by angle $\pi/8$ and scaling $2^{1/4}$ gives a rotation-dilation matrix with $a = 2^{1/4}\cos(\pi/8), b = 2^{1/4}\sin(\pi/8)$. The matrix is

$$A = 2^{1/4} \begin{bmatrix} \cos(\pi/8) & -\sin(\pi/8) \\ \sin(\pi/8) & \cos(\pi/8) \end{bmatrix} .$$

Problem 4) (10 points)

Let A be a 3×3 matrix such that $A^2 = 0$. That is, the product of A with itself is the zero matrix.

a) Verify that $\mathrm{Im}(A)$ is a subspace of $\ker(A)$.

b) Can $\mathrm{ran}(A) = 2$? If yes, give an example.

c) Can $\mathrm{ran}(A) = 1$? If yes, give an example.

d) Can $\mathrm{ran}(A) = 0$? If yes, give an example.

Solution:
a) If y is in the image, then $y = A(x)$ and $A(y) = A^2x = 0$.
b) No: If $\dim(\mathrm{ran}(A)) = 2$, then $\dim(\ker(A)) = 1$ and $\dim(\ker(A^2)$ has maximal 2 dimensions so that the rank of A^2 would be at least 1.
c) Yes: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
d) Yes: $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 5) (10 points)

Let b, c be arbitrary numbers. Consider the matrix $A = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -c \\ -b & c & 0 \end{bmatrix}$.

a) Find rref(A) and find a basis for the kernel and the image of A.

b) For which b, c is the kernel one dimensional?

c) Can the kernel be two dimensional?

Solution:

a) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & -b \\ 0 & 0 & 0 \end{bmatrix}$.

b) The kernel is always one-dimensional.

c) No. If the kernel had 2 or more dimensions, this would contradict the dimension formula $\dim(\ker)(A) + \dim(\text{im})(A) = 3$.

Problem 6) (10 points)

Consider the matrix $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$.

a) Use a series of elementary Gauss-Jordan row operations to find the reduced row echelon form rref(A) of A. Do only one elementary operations at each step.

b) Find the rank of A.

c) Find a basis for the image of A.

d) Find a basis for the kernel of A.

Solution:

a) We end up with $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) The rank is 2, then number of pivot columns.

c) A basis of the image are the first two column vectors in A.

d) A basis of the kernel is $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Problem 7) (10 points)

Let A be a 2×2 matrix and $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. We know that $B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find A^{2003}.

Hint. Write $B = (I_2 + C)$, note that $C^2 = 0$ and remember $(1+x)^n = 1 + nx + .. + x^n$.

Solution:

$B^{2003} = (1 + 2003C)$ so that $A^{2003} = S(1 + 2003\ C)S^{-1} = 1 + 2003\ (SCS^{-1})$. Because $S^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ and $2003SCS^{-1} = 2003 \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}$ we have - would have been easier to add up 2000 year ago... - $A^{2003} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2003 \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} = \begin{bmatrix} -12017 & 8012 \\ -18027 & 12019 \end{bmatrix}$.

Problem 8) (10 points)

Let A be a 5×5 matrix. Suppose a finite number of elementary row operations reduces A to the following matrix $B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

a) Find a basis of the kernel of A.

b) Suppose the elementary row operations used in reducing A to B are the following:

i) Add row 2 to row 3.
ii) Swap row 2 and row 4.
iii) Multiple row 4 by 1/2.
iv) Subtract row 1 from row 5.

Find a basis of the image of A.

Solution:
a) The matrix B and the matrix A have the same reduced row echelon form.
$\text{rref}(A) = \text{rref}(B) = B = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. This corresponds to $v = 0, w = t, z = 0, y = 0, x = t$ so that $t(1, 0, 0, 1, 0)$ is in the kernel.
b) We reverse the steps:
iv inverse) Add row 1 to row 5
iii inverse) multiply row 4 by 2
ii inverse) swap row 2 and row 4
i inverse) subtract row 2 from row 3

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Pick columns 1,2,3,5 of the last matrix.

Problem 9) (10 points)

a) Find a basis for the plane $x + 2y + z = 0$ in $\mathbf{R}^3$.

b) Find a 3×3 matrix which represents (with respect to the standard basis) a linear transformation with image the plane $x + 2y + z = 0$ and with the kernel the line $x = y = z$.

Solution:
a) $z = t, y = s, x = -2s - t$ gives $v_1 = [-1, 0, 1], v_2 = [0, -1/2, 1]$.
b) Take v_1, v_2 in the first columns. Then $-x + 0y + A_{13}z = 0$, $-1/2y + A_{23}z = 0$, $x + y + A_{33}z = 0$ for $x = y = z = 1$. Therefore $A_{13} = 1, A_{23} = 1/2, A_{33} = -2$. The matrix is $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1/2 & 1/2 \\ 1 & 1 & -2 \end{bmatrix}$.