10/22/2003 FIRST HOURLY WITH SOLUTIONS **Math 21b, Fall 2003**

Name:

MWF10 Izzet Coskun
MWF11 Oliver Knill

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T | **F**

If A, B are given $n \times n$ matrices, then the formula $(A-B)(A+B) = A^2 - B^2$ holds.

Solution:
Matrix multiplication is not commutative.

T | F

The rank of a diagonal matrix equals the number of non-zero entries.

Solution:
A diagonal matrix can easily put into row reduced echelon form by scaling the nonzero entries. These nonzero entries become leading 1.

T | F

The composition of a shear, a reflection, and a rotation in the plane is invertible.

Solution:
Each of these transformations are invertible. So is their composition.

T | F

If for a $n \times n$ matrix $A^{67} = I_n$, where I_n is the identity matrix, then A is invertible.

Solution:
If A were not invertible, it had a nontrivial $\vec{v}$ in the kernel and $A\vec{v} = \vec{0}$ implies $A^{67}\vec{v} = \vec{0}$ which would mean A^{67} were not invertible.

T | **F**

There exists a linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^2$ for which $T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Solution:
The three conditions together violate linearity. Note that the vectors in the two last equations add up to the vector in the first equation.

T | **F**

If A is a $n \times n$ matrix such that $A^3 = I_n$, then A is the identity matrix I_n.

Solution:
A could be a rotation by $2\pi/3$ for example.

T (highlighted)	F

There exists an invertible 3×3 matrix A such that 4 of its entries are 0.

Solution:
An example is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

T (highlighted)	F

It can happen that in rref(A) there are rows for which all entries are nonzero.

Solution:
Yes, $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is an example. Note that for $m \times n$ matrices with $m > 1$, this is not possible.

T (highlighted)	F

It can happen that in rref(A), there are columns for which all entries are nonzero.

Solution:
Yes, $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is an example. An other example is $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

T (highlighted)	F

The rank of a 3×4 matrix can be at most 3.

Solution:
True, look at the row reduced echelon form. There can be maximally 3 leading 1.

T	F (highlighted)

The rank of a 4×3 matrix may be 4.

Solution:
Again, look at the row reduced echelon form. There can be maximally 3 leading 1.

T (marked)	F	If A, B are $n \times n$ matrices such that AB is invertible, then A is invertible.

Solution:
Yes, $(AB)^{-1} = B^{-1}A^{-1}$ gives the inverse $A^{-1} = B(AB)^{-1}$.

T (marked)	F	If A, B are invertible $n \times n$ matrices, then AB is similar to BA.

Solution:
We need to find S such that $S^{-1}ABS = BA$. Just take $S = A$.

T (marked)	F	The number of leading ones in $\text{rref}(A)$ is the dimension of the image of A.

Solution:
That is a basic property of the rank = dimension of the image.

T	F (marked)	There exists a linear transformation whose image consists of exactly 3 distinct points.

Solution:
If P, Q, R are the three points, any linear combination of $P - Q$ and $R - Q$ is also in the image.

T	F (marked)	If T is a linear transformation then $\ker(T^7)$ is contained in $\ker(T)$.

Solution:
It is the other way round. If $\vec{v}$ is in the kernel of T then $\vec{v}$ is in the kernel of T^7. There are transformations where $T^7(\vec{x}) = \vec{0}$ for all $\vec{c}$ but T is not the zero matrix. Take for example $T(x) = S(x) - x$, where S is the shear.

T	F (marked)	If A and B are invertible $n \times n$ matrices, then so is $A + B$.

Solution:
False, take for example $B = -A$ where A is invertible.

T	F	There exists a linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^3$ for which $\ker(T) = \mathrm{im}(T)$.

Solution:
The dimensions of the image and the kernel would be the same. The dimension formula would show that $n = 3$ is an even number.

T	F	There exists a linear transformation T from $\mathbf{R}^4$ to $\mathbf{R}^4$ for which $\ker(T) = \mathrm{im}(T)$.

Solution:
Yes, you have seen an example in $\mathbf{R}^2$, it was $T(x) = S(x) - x$ where S is the shear. The transformation $R(x, y) = (T(x), T(y))$, where (x, y) are in $\mathbf{R}^2$ is a linear transformation on $\mathbf{R}^4$ and the image is equal to kernel. The matrix belonging to this transformation is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 2) (10 points)

Match each of matrices with one of the geometric descriptions below. You don't have to give explanations.

Matrix	Enter A-H here.
a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
b) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

Matrix	Enter A-H here.
e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	
f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
g) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
h) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

A) Shear along a plane.
B) Projection onto a plane.
C) Rotation around an axes.
D) Reflection at a point.
E) Projection onto a line.
F) Reflection at a plane.
G) Reflection at a line.
H) Identity transformation.

Solution:
a) = B)
b) = C)
c) = D)
d) = F)
e) = A)
f) = H)
g) = E)
h) = G)

Problem 3) (10 points)

Which of the following sets are linear spaces? You have to give explanations.

1) The image of the transformation $\mathbf{R}^2$ to $\mathbf{R}^2$ given by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2) The kernel of the projection from $\mathbf{R}^3$ to $\mathbf{R}^2$.

3) The solutions of the equation $2x + 3y - 5z = 12$ in $\mathbf{R}^3$.

4) All the points (x, y, z) in $\mathbf{R}^3$ which satisfy $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$.

Solution:
a) It is a nonlinear map, but the image is a linear subspace, namely $\mathbf{R}^2$.
b) The kernel is alwyas a linear space.
c) The solution space does not contain 0, so it can't be a linear subspace.
d) The solution is the set $\{x^2 + y^2 + z^2 = 0\}$ which is a linear subspace of $\mathbf{R}^3$.

Problem 4) (10 points)

In both problems, you have to find the solutions using Gauss-Jordan elimination.

a) (5 points) Find all solutions of $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 7 \end{bmatrix} .$$

Solution:
$x = 1/2, y = 1/2, z = -1$.

b) (5 points) Find all solutions to

$$\begin{aligned} x + y - 2z + 3w &= 4 \\ 2x + 3y + 3z - w &= 3 \\ 5x + 7y + 4z + w &= 10 \end{aligned}$$

Solution:

There are infinitely many solutions. One of the solutions is $\begin{bmatrix} 9 \\ -5 \\ 0 \\ 0 \end{bmatrix}$. We can add elements of the kernel of the coefficient matrix A and still have solutions.

Since $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -9 & 10 \\ 0 & 1 & 7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the kernel is $w = t, z = s, y = 7t - 7s, x = -10t + 9s$

so that a general element in the kernel is $s \begin{bmatrix} 9 \\ -7 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix}$.

A general solution is $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 9 \\ -7 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -10 \\ 7 \\ 0 \\ 1 \end{bmatrix}$.

Problem 5) (10 points)

a) (2 points) Find the 2×2 matrix A which defines the reflection S at the line $x = y$.

b) (2 points) Find the 2×2 matrix B which defines the counter clockwise rotation T with angle $60^\circ = \pi/3$ in the plane.

c) (2 points) Find the matrix $C = BA$ which belongs to the transformation $T \circ S$ which maps x to $T(S(x))$, where we first reflect and the rotate. What does this transformation do geometrically?

d) (2 points) Find the matrix C^{-1} which belongs to the inverse of the transformation $T \circ S$.

e) (2 points) Write C^{-1} as a product $C^{-1} = DR$ of a reflection R and a rotation D.

Solution:

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b) $B = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

c) $BA = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$. It is again a reflection at a line. To see that algebraically, realize that it is of the form $\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$. Geometrically, a rotation with angle α can be realized as the composition of two reflections at lines which form an angle $\alpha/2$. Doing this with two lines, where one of them is the line $x = y$ shows that BA is a reflection at a line which forms the angle $\pi/4 + \pi/6$ with the x-axes.

d) $C^{-1} = (BA)^{-1} = A^{-1}B^{-1}$. Because C is a reflection, also $C^{-1} = C$ is a reflection.

e) Because $C = BA$, we have $C^{-1} = A^{-1}B^{-1}$. Now $A^{-1} = A$ is a reflection, B^{-1} is a rotation by $-\pi/3$.

Problem 6) (10 points)

The matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ defines a linear transformation in the standard basis. How does the matrix to this transformation look like in the basis $\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$?

Solution:

Define $S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Now $B = S^{-1}AS = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Problem 7) (10 points)

Is the linear transformation $T(x) = A(x)$ with

$$A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

a shear? If so, find the line along which it is centered. (It is enough to give a nonzero vector $\vec{v}$ in that line.)

Solution:
Check that $T(\vec{x}) - \vec{x}$ is parallel to a fixed vector. It is enough to check that for two vectors: form $T\vec{e}_1 - \vec{e}_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $T\vec{e}_2 - \vec{e}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Because both vectors are parallel to $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we know also that linear combinations of $\vec{e}_1$ and $\vec{e_2}$ are parallel to $\vec{v}$. The transformation is a shear.

Problem 8) (10 points)

Find a basis for the kernel and the image of the following matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ and describe the kernel and the image geometrically.

Solution:
$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The image is two dimensional and spanned by the two first column vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ of A. The kernel is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The image is a plane, the kernel is the z-axes.

Problem 9) (10 points)

If A is a 4×5 matrix of rank 4. In each of the following questions, we need not only the answer but also a short explanation.

a) (2 points) Can the linear map $T(\vec{x}) = A\vec{x}$ be invertible ?

b) (3 points) What is the dimension of the image?

c) (3 points) What is the dimension of the kernel?

d) (2 points) How many solutions will the equation $A\vec{x} = \vec{b}$ have?

Solution:
a) No, the matrix is not a square matrix
b) The image is four dimensional.
c) The kernel is one dimensional by the dimension formula.
d) Infinitely many. Given a solution, we can add an element in the kernel and get a new solution.