

The EIGENVECTORS AND EIGENVALUES of a matrix  $A$  reveal the structure of  $A$ . Diagonalization in general eases the computations with  $A$ . It allows to find explicit formulas for LINEAR DYNAMICAL SYSTEMS  $x \mapsto Ax$ . Such systems are important for example in probability theory. The dot product leads to the notion of ORTHOGONALITY, allows measurements of angles and lengths and leads to geometric notations like rotation, reflection or projection in arbitrary dimensions. Least square solutions of  $Ax = b$  allow for example to solve fitting problems. DETERMINANTS of matrices appear in the definition of the characteristic polynomial and as volumes of parallelepipeds or as scaling values in change of variables. The notion allows to give explicit formulas for the inverse of a matrix or to solutions of  $Ax = b$ .

**ORTHOGONAL**  $\vec{v} \cdot \vec{w} = 0$ .

**LENGTH**  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$ .

**UNIT VECTOR**  $\vec{v}$  with  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$ .

**ORTHOGONAL SET**  $v_1, \dots, v_n$ : pairwise orthogonal.

**ORTHONORMAL SET** orthogonal and length 1.

**ORTHONORMAL BASIS** A basis which is orthonormal.

**ORTHOGONAL TO V**  $v$  is orthogonal to  $V$  if  $v \cdot x = 0$  for all  $x \in V$ .

**ORTHOGONAL COMPLEMENT OF V** Linear space  $V^\perp = \{v | v \text{ orthogonal to } V\}$ .

**PROJECTION ONTO V** orth. basis  $v_1, \dots, v_n$  in  $V$ ,  $\text{perp}_V(x) = (v_1 \cdot x)v_1 + \dots + (v_n \cdot x)v_n$ .

**GRAMM-SCHMIDT** Recursive  $u_i = v_i - \text{proj}_{V_{i-1}} v_i$ ,  $w_i = u_i / \|u_i\|$  leads to orthonormal basis.

**QR-FACTORIZATION**  $Q = [w_1 \cdots w_n]$ ,  $R_{ii} = u_i$ ,  $[R]_{ij} = w_i \cdot v_j, j > i$ .

**TRANSPOSE**  $[A^T]_{ij} = A_{ji}$ . Transposition switches rows and columns.

**SYMMETRIC**  $A^T = A$ .

**SKEWSYMMETRIC**  $A^T = -A$  ( $\Rightarrow R = e^A$  orthogonal:  $R^T = e^{A^T} = e^{-A} = R^{-1}$ ).

**DOT PRODUCT AS MATRIX PRODUCT**  $v \cdot w = v^T \cdot w$ .

**ORTHOGONAL MATRIX**  $Q^T Q = 1$ .

**ORTHOGONAL PROJECTION** onto  $V$  is  $AA^T$ , columns  $\vec{v}_i$  are orthonormal basis in  $V$ .

**ORTHOGONAL PROJECTION** onto  $V$  is  $A(A^T A)^{-1} A^T$ , columns  $\vec{v}_i$  are basis in  $V$ .

**NORMAL EQUATION** to  $Ax = b$  is the consistent system  $A^T Ax = A^T b$ .

**LEAST SQUARE SOLUTION** of  $A\vec{x} = \vec{b}$  is  $\vec{x}_* = (A^T A)^{-1} A^T \vec{b}$ .

**ANGLE** between two vectors  $x, y$  is  $\alpha = \arccos((x \cdot y) / (\|x\| \|y\|))$ .

**CORRELATION COEFFICIENT**  $(x \cdot y) / (\|x\| \|y\|)$  is  $\cos(\text{angle})$ .

**SCATTER PLOT** Visualization of data points  $(x_i, y_i)$  in the plane.

**DETERMINANT**  $\det(A) = (\sum_{\text{even } \pi} \pi - \sum_{\text{odd } \pi} \pi) A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$ .

**PARALLELEPIPED** Image of unit cube by  $A$ . Spanned by columns of  $A$ , volume  $\sqrt{|\det(A^T A)|}$ .

**MINOR**  $A_{ij}$ , the matrix with row  $i$  and column  $j$  deleted.

**CLASSICAL ADJOINT**  $\text{adj}(A)_{ij} = (-1)^{i+j} \det(A_{ji})$  (note switch of  $ij$ ).

**ORIENTATION**  $\text{sign}(\det(A))$  defines orientation of column vectors of  $A$ .

**TRACE** is  $\text{tr}(A) = \sum_i A_{ii}$ , the sum of diagonal elements of  $A$ .

**CHARACTERISTIC POLYNOMIAL**  $f_A(\lambda) = \det(\lambda - A) = \lambda^n - \text{tr}(A)\lambda + \dots + (-1)^n \det(A)$ .

**EIGENVALUES AND EIGENVECTORS**  $Av = \lambda v, v \neq 0$ , eigenvalue  $\lambda$ , eigenvector  $v$ .

**FACTORISATION OF  $f_A(\lambda)$**  Have  $f_A(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$  **with roots**  $\lambda_i$ .

**ALGEBRAIC MULTIPLICITY**  $k$  if  $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$  with  $g(\lambda_0) \neq 0$ .

**GEOMETRIC MULTIPLICITY** The dimension of the kernel of  $\lambda - A$ .

**KERNEL AND EIGENVECTORS** Vectors in the kernel of  $A$  are eigenvectors of  $A$ .

**EIGENBASIS** Basis which consists of eigenvectors of  $A$ .

**COMPLEX NUMBERS**  $z = x + iy = |z| \exp(i \arg(z)) = r e^{i\phi} = r \exp(i\phi) = r \cos(\phi) + ir \sin(\phi)$ .

**MODULUS AND ARGUMENT**  $|z| = |x + iy| = \sqrt{x^2 + y^2}$ ,  $\phi = \arg(z) = \arctan(y/x)$ .

**CONJUGATE**  $\bar{z} = x - iy$  if  $z = x + iy$ .

**LINEAR DYNAMICAL SYSTEM** Linear map  $x \mapsto Ax$  defines orbit  $\vec{x}(t+1) = A\vec{x}(t)$ .

### Skills checklist

COMPUTE DETERMINANTS IN DIFFERENT WAYS (row reduction, Laplace, volume, patterns).

GRAMM-SCHMIDT ORTHOGONALISATION (algorithm).

COMPUTING EIGENVALUES OF A (factoring characteristic polynomial).

COMPUTING EIGENVECTORS OF A (determining kernel of  $\lambda - A$ ).

COMPUTING ALGEBRAIC AND GEOMETRIC MULTIPLICITIES (know the definitions).

COMPUTING ORTHOGONAL PROJECTION ONTO LINEAR SUBSPACES (formula).

PRODUCE LEAST SQUARE SOLUTION OF LINEAR EQUATION (formula).

ALGEBRA OF MATRICES (multiplication, inverse, recall rank, image, kernel, inverse).

CALCULATION WITH COMPLEX NUMBERS (operations, roots, identifying  $\text{Re}(z)$ ,  $\text{Im}(z)$ ).

DIAGONALIZE MATRIX (find eigensystem, build  $S$  so that  $S^{-1}AS$  is diagonal).

DETERMINE STABILITY OF A LINEAR SYSTEM (eigenvalue criterion, especially 2D case).

ORTHOGONAL IMPLIES INDEPENDENT. Orthogonal vectors are linearly independent.

ORTHOGONAL PLUS SPAN IMPLIES BASIS.  $n$  orthogonal vectors  $\mathbf{R}^n$  form a basis.

ORTHOGONAL COMPLEMENT IS LINEAR SPACE. Notation  $V^\perp$ . Have  $(V^\perp)^\perp = V$ .

LINE TO PROJECTION IS ORTHOGONAL TO  $V$ .  $\vec{x} - \text{proj}_V(\vec{x})$  is orthogonal to  $V$ .

PYTHAGORAS:  $x, y$  orthogonal  $\Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

PROJECTION CONTRACTS  $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$ .

IMAGE OF  $A$  IS ORTHOGONAL COMPLEMENT TO KERNEL OF  $A^T$ .

DIMENSION OF ORTHOGONAL COMPLEMENT.  $\dim(V) + \dim(V^T) = n$ .

CAUCHY-SCHWARTZ INEQUALITY:  $|x \cdot y| \leq \|x\| \|y\|$ .

TRIANGLE INEQUALITY:  $\|x + y\| \leq \|x\| + \|y\|$ .

ROW VECTORS OF  $A$  are orthogonal to  $\ker(A)$ . Short  $\text{im}(A^T) = \ker(A)^\perp$ .

ORTHOGONAL TRANSFORMATIONS preserve angle, length. Columns form orthonormal basis.

ORTHOGONAL PROJECTION:  $P = A(A^T A)^{-1} A^T$  onto  $V$  with  $A = [v_1, \dots, v_n]$ ,  $V = \text{im}(A)$ .

ORTHOGONAL PROJECTION: onto  $V$  is  $AA^T$  if  $A = [v_1, \dots, v_n]$  is orthogonal.

ORTHOGONAL PROJECTIONS are not orthogonal transformations in general.

KERNEL OF  $A$  AND  $A^T A$  are the same:  $\ker(A) = \ker(A^T A)$ .

DETERMINANTS  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ .  $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - fha - bdi$ .

DETERMINANT OF DIAGONAL OR TRIANGULAR MATRIX: product of diagonal entries.

DETERMINANT OF BLOCK MATRIX:  $\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A)\det(B)$ .

PROPERTIES OF DETERMINANTS.  $\det(AB) = \det(A)\det(B)$ ,  $\det(A^{-1}) = 1/\det(A)$ .

PROPERTIES OF DETERMINANTS.  $\det(SAS^{-1}) = \det(A)$ ,  $\det(A^T) = \det(A)$ .

LINEARITY OF DETERMINANT.  $\det([A\vec{a}B]) + \det([A\vec{b}B]) = \det([A(\vec{a} + \vec{b})B])$ ,  $\det([A\lambda\vec{a}B]) = \lambda\det([A\vec{a}B])$ .

DETERMINANT OF  $[A]_{ij} = \vec{v}_i \cdot \vec{v}_j$  is square of determinant of  $B = [\vec{v}_1, \dots, \vec{v}_n]$ .

SWITCHING OF TWO ROWS.  $\det(B) = -\det(A)$ .

ADDING ROW TO GIVEN DIFFERENT ROW:  $\det(B) = \det(A)$

PARALLELEPIPED.  $|\det(A)| = \text{vol}(E)$  with  $E$ =parallelepiped spanned by columns of  $A$ .

$K$ -EPIPED.  $\sqrt{\det(A^T A)} = \text{vol}(k\text{-dimensional parallelepiped spanned by column vectors of } A)$ .

RREF.  $\det(A) = (-1)^s (\prod_i c_i) \det(\text{rref}(A))$  with  $c_i$  row scaling factors and  $s$  row switches.

IN ODD DIMENSIONS a real matrix has a real eigenvalue.

IN EVEN DIMENSIONS a real matrix with negative determinant has real eigenvalue.

PROPERTIES OF TRANSPOSE.  $(A^T)^T = A$ ,  $(AB)^T = B^T A^T$ ,  $(A^{-1})^T = (A^T)^{-1}$ .

DIAGONALISATION:  $A$   $n \times n$  matrix,  $S$  eigenvectors of  $A$  in columns,  $S^{-1}AS$  diagonal.

JORDAN NORMAL FORM: In the complex, every  $A$  can be brought into Jordan normal form.

NONTRIVIAL KERNEL  $\Leftrightarrow \det(A) = 0$ .

INVERTIBLE MATRIX  $\Leftrightarrow \det(A) \neq 0$ .

LAPLACE EXPANSION.  $\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \dots + (-1)^{i+n} a_{in} \det(A_{in})$

ORTHOGONAL MATRICES  $A$  have  $\det(A) = \pm 1$

ROTATIONS satisfy  $\det(A) = 1$  in all dimensions.

ROTATIONS with angle  $\phi$  in the plane have eigenvalues  $\exp(i\phi)$ .

QR DECOMPOSITION.  $A = QR$  orthogonal  $A$ , upper triangular  $R$ . Have  $|\det(A)| = \prod_{i=1}^n R_{ii}$ .

CRAMER'S RULE. Solve  $Ax = b$  by  $x_i = \det(A_i)/\det(A)$ , where  $A_i$  is  $A$  with  $b$  in column  $i$ .

CLASSICAL ADJOINT AND INVERSE.  $A^{-1} = \text{adj}(A)/\det(A)$ .

DETERMINANT IS PRODUCT OF EIGENVALUES.  $\det(A) = \prod_i \lambda_i$ .

TRACE IS SUM OF EIGENVALUES.  $\text{tr}(A) = \sum_i \lambda_i$ .

GEOMETRIC MULTIPLICITY  $\leq$  ALGEBRAIC MULTIPLICITY.

DIFFERENT EIGENVALUES  $\Rightarrow$  EIGENSYSTEM.  $\lambda_i \neq \lambda_j, i \neq j \Rightarrow$  eigenvectors form basis.

EIGENVALUES OF  $A^T$  agree with eigenvalues of  $A$  (same characteristic polynomial).

RANK OF  $A^T$  is equal to the rank of  $A$ .

REFLECTION at linear  $k$ -dimensional subspace in  $\mathbf{R}^n$  has determinant  $(-1)^{(n-k)}$ .

DE MOIVRE FORMULA:  $z^n = \exp(in\phi) = \cos(n\phi) + i \sin(n\phi) = (\cos(\phi) + i \sin(\phi))^n$ .

FUNDAMENTAL THEOREM OF ALGEBRA.  $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + \lambda_0$  has  $n$  roots.

NUMBER OF EIGENVALUES. A  $n \times n$  matrix has exactly  $n$  eigenvalues (count multiplicity).

POWER OF A MATRIX.  $A^n$  has eigenvalues  $\lambda^n$  if  $A$  has eigenvalue  $\lambda$ .

EIGENVALUES OF  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  are  $\lambda_{\pm} = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$ .

EIGENVECTORS OF  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  with  $c \neq 0$  are  $v_{\pm} = [\lambda_{\pm} - d, c]$ .

ROTATION-DILATION MATRIX:  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ , eigenvalues  $p \pm iq$ , eigenvectors  $(\pm i, 1)$ .

ROTATION-DILATION MATRIX: linear stable origin if and only if  $|\det(A)| < 1$ .

DATA FITTING: Least square solution of  $A\vec{x} = \vec{b}$ , where  $A$  and  $b$  depend on data and functions.