

## MATRIX PRODUCT

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HOMEWORK: Section 2.4: 4,14,28,40,76,48\*,66\*

MATRIX PRODUCT. If  $B$  is a  $p \times m$  matrix and  $A$  is a  $m \times n$  matrix, then  $BA$  is defined as the  $p \times n$  matrix with entries  $(BA)_{ij} = \sum_{k=1}^m B_{ik}A_{kj}$ .

EXAMPLE. If  $B$  is a  $3 \times 4$  matrix, and  $A$  is a  $4 \times 2$  matrix then  $BA$  is a  $3 \times 2$  matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

COMPOSING LINEAR TRANSFORMATIONS. If  $S: \mathbf{R}^n \rightarrow \mathbf{R}^m, x \mapsto Ax$  and  $T: \mathbf{R}^m \rightarrow \mathbf{R}^p, y \mapsto By$  are linear transformations, then their composition  $T \circ S: x \mapsto BA(x)$  is a linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^p$ . The corresponding matrix is the matrix product  $B \cdot A$ .

EXAMPLE. Find the matrix which is a composition of a rotation around the  $x$ -axes by an angle  $\pi/2$  followed by a rotation around the  $z$ -axes by an angle  $\pi/2$ .

SOLUTION. The first transformation has the property that  $e_1 \rightarrow e_1, e_2 \rightarrow e_3, e_3 \rightarrow -e_2$ , the second  $e_1 \rightarrow e_2, e_2 \rightarrow -e_1, e_3 \rightarrow e_3$ . If  $A$  is the matrix belonging to the first transformation and  $B$  the second, then  $BA$  is the matrix to the composition. The composition maps  $e_1 \rightarrow -e_2 \rightarrow e_3 \rightarrow e_1$  is a rotation around a long diagonal.  $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

EXAMPLE. A rotation dilation is the composition of a rotation by  $\alpha = \arctan(b/a)$  and a dilation (=scale) by  $r = \sqrt{a^2 + b^2}$ .

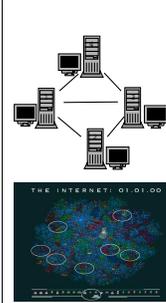
REMARK. Matrix multiplication is a generalization of usual multiplication of numbers or the dot product.

MATRIX ALGEBRA. Note that  $AB \neq BA$  in general! Otherwise, the same rules apply as for numbers:  $A(BC) = (AB)C, AA^{-1} = A^{-1}A = 1_n, (AB)^{-1} = B^{-1}A^{-1}, A(B+C) = AB + AC, (B+C)A = BA + CA$  etc.

PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If  $B$  is a  $m \times n$  matrix and  $A$  is a  $n \times p$  matrix, and assume the entries are  $k \times k$  matrices, then  $BA$  is a  $m \times p$  matrix where each entry  $(BA)_{ij} = \sum_{k=1}^n B_{ik}A_{kj}$  is a  $k \times k$  matrix. Partitioning matrices can be useful to improve the speed of matrix multiplication (i.e. Strassen algorithm).

EXAMPLE. If  $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ , where  $A_{ij}$  are  $k \times k$  matrices with the property that  $A_{11}$  and  $A_{22}$  are invertible, then  $B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$  is the inverse of  $A$ .

APPLICATIONS. (The material which follows is for motivation purposes only, more applications appear in the homework).



NETWORKS. Let us associate to the computer network (shown at the left) a

$$\text{matrix } \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

To a worm in the first computer we associate a vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

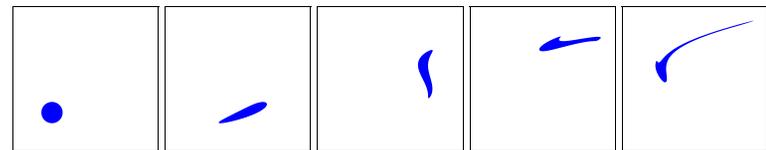
. The vector  $Ax$  has a 1 at the places, where the worm could be in the next step. The vector  $(AA)(x)$  tells, in how many ways the worm can go from the first computer to other hosts in 2 steps. In our case, it can go in three different ways back to the computer itself.

Matrices help to solve combinatorial problems (see movie "Good will hunting"). For example, what does  $[A^{100}]_{22}$  tell about the worm infection of the network? What does it mean if  $A^{100}$  has no zero entries?



FRACTALS. Closely related to linear maps are **affine maps**  $x \mapsto Ax + b$ . They are compositions of a linear map with a translation. It is **not** a linear map if  $B(0) \neq 0$ . Affine maps can be disguised as linear maps in the following way: let  $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$  and define the  $(n+1) \times (n+1)$  matrix  $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$ . Then  $By = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}$ .

Fractals can be constructed by taking for example 3 affine maps  $R, S, T$  which contract area. For a given object  $Y_0$  define  $Y_1 = R(Y_0) \cup S(Y_0) \cup T(Y_0)$  and recursively  $Y_k = R(Y_{k-1}) \cup S(Y_{k-1}) \cup T(Y_{k-1})$ . The above picture shows  $Y_k$  after some iterations. In the limit, for example if  $R(Y_0), S(Y_0)$  and  $T(Y_0)$  are disjoint, the sets  $Y_k$  converge to a **fractal**, an object with dimension strictly between 1 and 2.



CHAOS. Consider a map in the plane like  $T: \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 2 \sin(x) - y \\ x \end{bmatrix}$ . We apply this map again and again and follow the points  $(x_1, y_1) = T(x, y), (x_2, y_2) = T(T(x, y))$ , etc. One writes  $T^n$  for the  $n$ -th iteration of the map and  $(x_n, y_n)$  for the image of  $(x, y)$  under the map  $T^n$ . The linear approximation of the map at a point  $(x, y)$  is the matrix  $DT(x, y) = \begin{bmatrix} 2 + 2 \cos(x) - 1 & -1 \\ 1 & 0 \end{bmatrix}$ . (If  $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$ , then the row vectors of  $DT(x, y)$  are just the gradients of  $f$  and  $g$ ).  $T$  is called **chaotic at**  $(x, y)$ , if the entries of  $D(T^n)(x, y)$  grow exponentially fast with  $n$ . By the **chain rule**,  $D(T^n)$  is the product of matrices  $DT(x_i, y_i)$ . For example,  $T$  is chaotic at  $(0, 0)$ . If there is a positive probability to hit a chaotic point, then  $T$  is called chaotic.

FALSE COLORS. Any color can be represented as a vector  $(r, g, b)$ , where  $r \in [0, 1]$  is the red  $g \in [0, 1]$  is the green and  $b \in [0, 1]$  is the blue component. Changing colors in a picture means applying a transformation on the cube. Let  $T : (r, g, b) \mapsto (g, b, r)$  and  $S : (r, g, b) \mapsto (r, g, 0)$ . What is the composition of these two linear maps?



OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray  $y(s) = x + ms$  in the plane is described by a vector  $(x, m)$ . Following the light ray over a distance of length  $L$  corresponds to the map  $(x, m) \mapsto (x + mL, m)$ . In the lens, the ray is bent depending on the height  $x$ . The transformation in the lens is  $(x, m) \mapsto (x, m - kx)$ , where  $k$  is the strength of the lens.



$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:

- 1) Eye looking far:  $A_R B_k$ .
- 2) Eye looking at distance  $L$ :  $A_R B_k A_L$ .
- 3) Telescope:  $B_{k_2} A_L B_{k_1}$ . (More about it in problem 80 in section 2.4).