

HEAT AND WAVE EQUATION

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NOTATION. We write $T_x(x, t)$ and $T_t(x, t)$ for the partial derivatives with respect to x or t .

THE HEAT EQUATION. The temperature distribution $T(x, t)$ in a metal bar $[0, T]$ satisfies the **heat equation** $T_t(x, t) = \mu T_{xx}(x, t)$. At every point, the rate of change of the temperature is proportional to the second derivative of $T(x, t)$ at x . The function $T(x, t)$ is zero at both ends of the bar $T(0, t) = T(\pi, t) = 0$ and $f(x) = T(x, 0)$ is a given initial temperature distribution.

SOLVING IT. This **partial differential equation** (PDE) is solved by writing $T(x, t) = u(x)v(t)$ which leads to $\dot{v}(t)u(x) = v(t)u''(x)$ or $\dot{v}(t)/(\mu v(t)) = u''(x)/u(x)$. Because the LHS does not depend on x and the RHS not on t , this must be a constant λ . The ODE $u''(x) = \lambda u(x)$ is an eigenvalue problem $D^2 u = \lambda u$ which has only solutions when $\lambda = -k^2$ for integers k . The eigenfunctions of D^2 are $u_k(x) = \sin(kx)$. The second equation $\dot{v}(t) = k^2 \mu v$ is solved by $v(t) = e^{-k^2 \mu t}$ so that $u(x, t) = \sin(kx)e^{-k^2 t}$ is a solution of the heat equation. Because linear combinations of solutions are solutions too, a general solution is of the form $T(x, t) = \sum_{k=1}^{\infty} a_k \sin(kx)e^{-k^2 t}$. For fixed t this is a **Fourier series**. The coefficients a_k are determined from $f(x) = T(x, 0) = \sum_{k=1}^{\infty} a_k \sin(kx)$, where $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)$. (The factor 2 comes from the fact that f should be thought to be extended to an odd function on $[-\pi, \pi]$ and the integral from $[-\pi, 0]$ is the same as the from $[0, \pi]$.)

FOURIER: (Solving the heat equation was the birth of Fourier theory)
 The heat equation $T_t(x, t) = \mu T_{xx}(x, t)$ with smooth $T(x, 0) = f(x), T(0, 0) = T(\pi, 0) = 0$ has the solution $\sum_{k=1}^{\infty} a_k \sin(kx)e^{-k^2 \mu t}$ with $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx$.

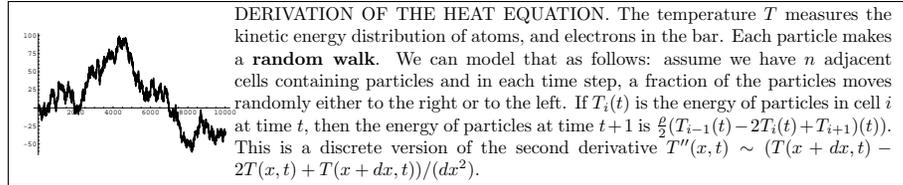
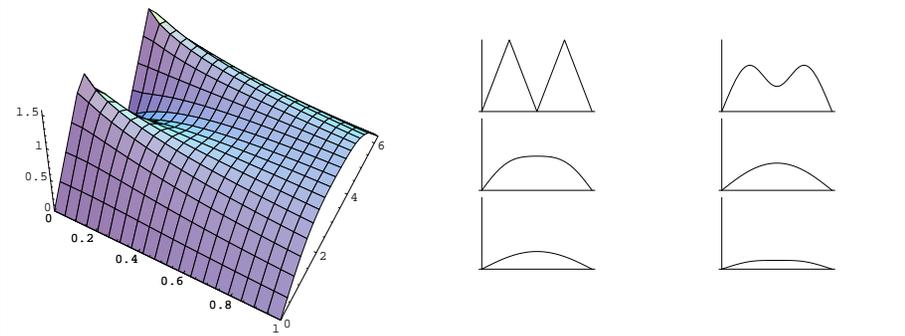
Remark. If $T(0, 0) = a, T(\pi, 0) = b$ instead, then the solution is $T(x, t) = T_0(x, t) + a + bx/\pi$, where $T_0(x, t)$ is the solution in the box using $f_0(x) = f(x) - (a + bx/\pi)$ which satisfies $f_0(0) = f_0(\pi) = 0$. (Because $T(x, t) = a + bx/\pi$ is a solution of the heat equation, we can add it to any solution and get new solutions. This allows to tune the boundary conditions).

EXAMPLE. $f(x) = x$ on $[-\pi/2, \pi/2]$ π -periodically continued has the Fourier coefficients $a_k = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(kx) dx = 4/(k^2 \pi)(-1)^{(k-1)/2}$ for odd k and 0 else. The solution is

$$T(x, t) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{(2m+1)^2 \pi} e^{-\mu(2m+1)^2 t} \sin((2m+1)x).$$

The exponential term containing the time makes the function $T(x, t)$ converge to 0: "The bar cools." The higher frequency terms are damped faster because "Smaller disturbances are smoothed out faster."

VISUALIZATION. We can just plot the graph of the function $T(x, t)$ or plot the temperature distribution for different times t_i .



FOURIER TRANSFORM=DIAGONALISATION OF D. Fourier theory actually **diagonalizes** the linear map D : write $f = \sum_n \hat{f}_n e^{inx} = S^{-1}f$ and $Sf = \hat{f} = (\dots, c_{-1}, c_0, c_1, \dots)$. $SDS^{-1}\hat{f} = SD\sum_n \hat{f}_n e^{inx} = S\sum_n n\hat{f}_n e^{inx} = (\dots, i(-1)c_{-1}, 0c_0, ic_1, \dots)$. This means that differentiation changes the Fourier coefficients as $c_n \mapsto inc_n$. From this we see that the diagonalisation of D^2 is multiplying with $-n^2$. Now, if $\dot{u} = D^2 u$, then $\dot{\hat{u}}_n = -\mu n^2 \hat{u}_n$ so that $\hat{u}_n(t) = e^{-\mu n^2 t} \hat{u}_n(0)$. The partial differential equation is transformed into a sequence of ordinary differential equations.

Differentiation of the function $f \xrightarrow{S} \hat{f}$ $D \downarrow$ $\downarrow M$ $Df \xrightarrow{S^{-1}} M\hat{f}$ becomes multiplication with the diagonal matrix $M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2i & 0 & 0 & 0 & 0 \\ \cdot & 0 & i & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & i & 0 \\ \cdot & 0 & 0 & 0 & 0 & 2i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$

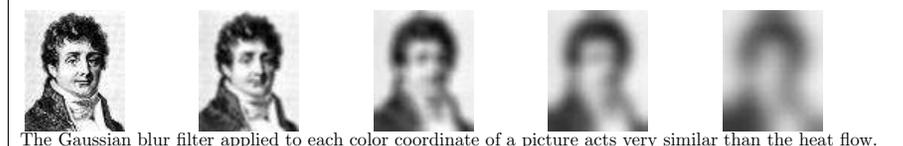
WAVE EQUATION. The wave equation $T_{tt}(x, t) = c^2 T_{xx}(x, t)$ is solved in a similar way as the heat equation: writing $T(x, t) = u(x)v(t)$ gives $\ddot{v}u = c^2 v u''$ or $\ddot{v}/(c^2 v) = u''/u$. Because the left hand is independent of x and the right hand side is independent of t , we have $\ddot{v}/(c^2 v) = u''/u = -k^2 = const$. The right equation has solutions $u_k(x) = \sin(kx)$. Now, $v_k(t) = a_k \cos(kct) + b_k \sin(kct)$ solves $\ddot{v} = -c^2 k^2 v$ so that $T_k(x, t) = u_k(x)v_k(t) = \sin(kx)(a_k \cos(kct) + b_k \sin(kct))$ is a solution of the wave equation. General solutions can be obtained by taking **superpositions of these waves** $T(x, t) = \sum_{k=1}^{\infty} a_k \sin(kx) \cos(kct) + b_k \sin(kx) \sin(kct)$. The coefficients a_k, b_k are obtained from $T(x, 0) = \sum_{k=1}^{\infty} a_k \sin(kx)$ and $\dot{T}(x, 0) = \sum_{k=1}^{\infty} kcb_k \sin(kx)$. Unlike in the heat equation both $T(x, 0)$ and $\dot{T}(x, 0)$ have to be specified.

WAVE EQUATION: The wave equation $T_{tt} = c^2 T_{xx}$ with smooth $T(x, 0) = f(x), \dot{T}(x, 0) = g(x), T(0, 0) = T(\pi, 0) = 0$ has the solution $\sum_{k=1}^{\infty} a_k \sin(kx) \cos(kct) + b_k \sin(kx) \sin(kct)$ with $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx$ and $kcb_k = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(kx) dx$.

QUANTUM MECHANICS = MATRIX MECHANICS A particle in a box is described by a linear map $L = -D^2 \hbar^2 / (2m)$ acting on functions on $[0, \pi]$. In the Fourier picture, this linear map is $M = \frac{\hbar^2}{2m} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -4 & 0 & 0 & 0 \\ \cdot & 0 & -1 & 0 & 0 \\ \cdot & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & -1 \\ \cdot & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & 0 & -4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$ The diagonal entries are the eigenvalues of L as usual after diagonalisation.

Quantum mech. observables are symmetric linear maps: for example: position, $X : f \mapsto xf$, momentum $P : f \mapsto i\hbar \nabla f$, kinetic energy $f \mapsto P^2/2m = -\hbar^2 \Delta / (2m)$, Coulomb potential energy $f \mapsto (4\pi\epsilon_0)^{-1} e^2 / r f$.

HIGHER DIMENSIONS. The heat or wave equations can be solved also in **higher dimensions** using Fourier theory. For example, on a metallic plate $[-\pi, \pi] \times [-\pi, \pi]$ the temperature distribution $T(x, y, t)$ satisfies $T_t = \mu(T_{xx} + T_{yy})$. The Fourier series of a function $f(x, y)$ in two variables is $\sum_{n, m} c_{n, m} e^{inx+my}$, where $c_{n, m} = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) e^{-inx} e^{-imy} dx dy$.



The Gaussian blur filter applied to each color coordinate of a picture acts very similar than the heat flow.