

COMPLEX LINEAR 1D CASE.  $\dot{x} = \lambda x$  for  $\lambda = a + ib$  has solution  $x(t) = e^{at} e^{ibt} x(0)$  and norm  $\|x(t)\| = e^{at} \|x(0)\|$ .

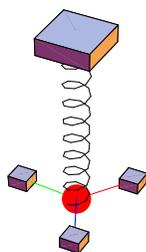
OSCILLATOR: The system  $\ddot{x} = -\lambda x$  has the solution  $x(t) = \cos(\sqrt{\lambda}t)x(0) + \sin(\sqrt{\lambda}t)\dot{x}(0)/\sqrt{\lambda}$ .

DERIVATION.  $\dot{x} = y, \dot{y} = -\lambda x$  and in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

and because  $A$  has eigenvalues  $\pm i\sqrt{\lambda}$ , the new coordinates move as  $a(t) = e^{i\sqrt{\lambda}t}a(0)$  and  $b(t) = e^{-i\sqrt{\lambda}t}b(0)$ .

Writing this in the original coordinates  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$  and fixing the constants gives  $x(t), y(t)$ .



EXAMPLE. THE SPINNER. The spinner is a rigid body attached to a spring aligned around the z-axis. The body can rotate around the z-axis and bounce up and down. The two motions are coupled in the following way: when the spinner winds up in the same direction as the spring, the spring gets tightend and the body gets a lift. If the spinner winds up to the other direction, the spring becomes more relaxed and the body is lowered. Instead of reducing the system to a 4D first order system, system  $\frac{d}{dt}\vec{x} = A\vec{x}$ , we will keep the second time derivative and diagonalize the 2D system  $\frac{d^2}{dt^2}\vec{x} = A\vec{x}$ , where we know how to solve the one dimensional case  $\frac{d^2}{dt^2}v = -\lambda v$  as  $v(t) = A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t)$  with constants  $A, B$  depending on the initial conditions,  $v(0), \dot{v}(0)$ .

SETTING UP THE DIFFERENTIAL EQUATION.

$x$  is the angle and  $y$  the height of the body. We put the coordinate system so that  $y = 0$  is the point, where the body stays at rest if  $x = 0$ . We assume that if the spring is winded up with an angle  $x$ , this produces an upwards force  $x$  and a momentum force  $-3x$ . We furthermore assume that if the body is at position  $y$ , then this produces a momentum  $y$  onto the body and an upwards force  $y$ . The differential equations

$$\begin{aligned} \ddot{x} &= -3x + y \\ \ddot{y} &= -y + x \end{aligned} \quad \text{can be written as } \ddot{v} = Av = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} v.$$

FINDING GOOD COORDINATES  $w = S^{-1}v$  is obtained with getting the eigenvalues and eigenvectors of  $A$ :  $\lambda_1 = -2 - \sqrt{2}, \lambda_2 = -2 + \sqrt{2}$   $v_1 = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$  so that  $S = \begin{bmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{bmatrix}$ .

SOLVE THE SYSTEM  $\ddot{a} = \lambda_1 a, \ddot{b} = \lambda_2 b$  IN THE GOOD COORDINATES  $\begin{bmatrix} a \\ b \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$ .

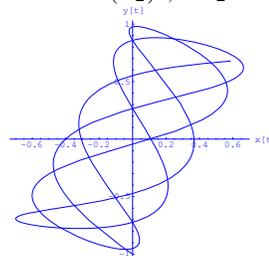
$$a(t) = A \cos(\omega_1 t) + B \sin(\omega_1 t), \quad \omega_1 = \sqrt{-\lambda_1}, \quad b(t) = C \cos(\omega_2 t) + D \sin(\omega_2 t), \quad \omega_2 = \sqrt{-\lambda_2}.$$

THE SOLUTION IN THE ORIGINAL COORDINATES.  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} =$

$S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ . At  $t = 0$  we know  $x(0), y(0), \dot{x}(0), \dot{y}(0)$ . This fixes the constants

in  $x(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)$

$+ A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)$ . The curve  $(x(t), y(t))$  traces a Lyssajoux curve:



ASYMPTOTIC STABILITY.

A linear system  $\dot{x} = Ax$  in the 2D plane is asymptotically stable if and only if  $\det(A) > 0$  and  $\text{tr}(A) < 0$ .

PROOF. If the eigenvalues  $\lambda_1, \lambda_2$  of  $A$  are real then both being negative is equivalent with  $\lambda_1 \lambda_2 = \det(A) > 0$  and  $\text{tr}(A) = \lambda_1 + \lambda_2 < 0$ . If  $\lambda_1 = a + ib, \lambda_2 = a - ib$ , then a negative  $a$  is equivalent to  $\lambda_1 + \lambda_2 = 2a < 0$  and  $\lambda_1 \lambda_2 = a^2 + b^2 > 0$ .

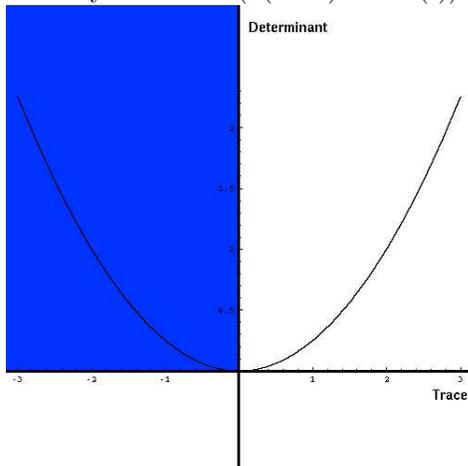
ASYMPTOTIC STABILITY COMPARISON OF DISCRETE AND CONTINUOUS SITUATION.

The trace and the determinant are independent of the basis, they can be computed fast, and are real if  $A$  is real. It is therefore convenient to determine the region in the  $\text{tr} - \text{det}$ -plane, where continuous or discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related to a discrete system, it is important not to mix these two situations up.

Continuous dynamical system.

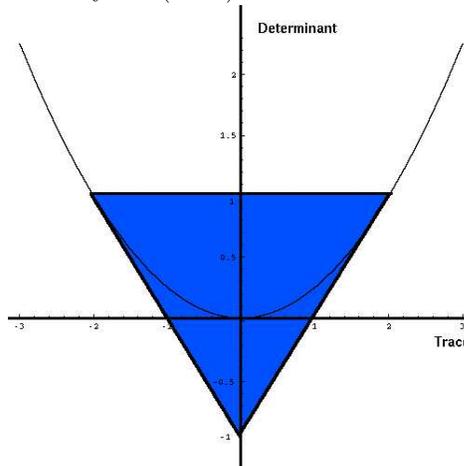
Discrete dynamical system.

Stability of  $\dot{x} = Ax$  ( $x(t+1) = e^A x(t)$ ).



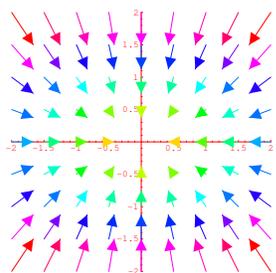
Stability in  $\det(A) > 0, \text{tr}(A) > 0$   
 Stability if  $\text{Re}(\lambda_1) < 0, \text{Re}(\lambda_2) < 0$ .

Stability of  $x(t+1) = Ax$

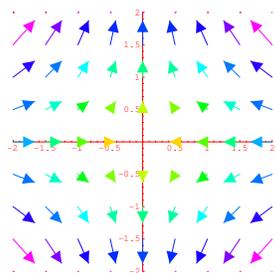


Stability in  $|\text{tr}(A)| < 1, -1 < \det(A) < 1$   
 Stability if  $|\lambda_1| < 1, |\lambda_2| < 1$ .

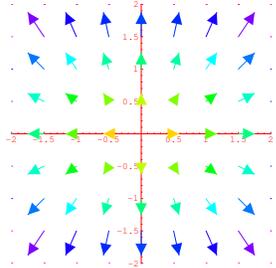
PHASEPORTRAITS. (In two dimensions we can plot the vector field, draw some trajectories)



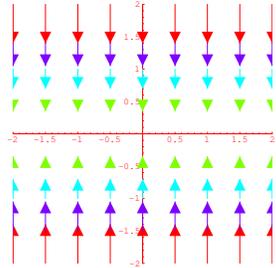
$\lambda_1 < 0$   
 $\lambda_2 < 0$ ,  
 i.e  $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$



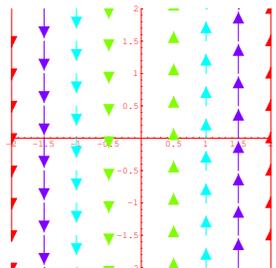
$\lambda_1 < 0$   
 $\lambda_2 > 0$ ,  
 i.e  $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$



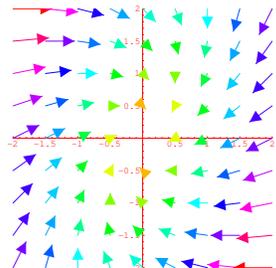
$\lambda_1 > 0$   
 $\lambda_2 > 0$ ,  
 i.e  $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



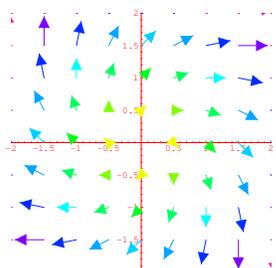
$\lambda_1 = 0$   
 $\lambda_2 < 0$ ,  
 i.e  $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$



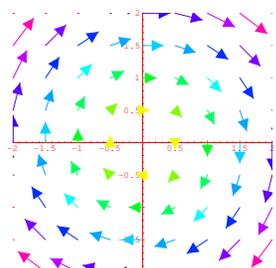
$\lambda_1 = 0$   
 $\lambda_2 = 0$ ,  
 i.e  $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a < 0$   
 $\lambda_2 = a - ib$ ,  
 i.e  $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a > 0$   
 $\lambda_2 = a - ib$ ,  
 i.e  $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = ib$   
 $\lambda_2 = -ib$ ,  
 i.e  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$