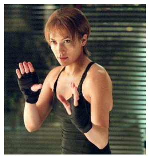
DIMENSION **Math 21b, O. Knill**

REVIEW LINEAR SUBSPACE $X \subset R^n$ is a **linear space** if $\vec{0} \in X$ and if X is closed under addition and scalar multiplication. Examples are $\mathbf{R}^n, X = \ker(A), X = \mathrm{im}(A)$, or the row space of a matrix. In order to describe linear spaces, we had the notion of a basis:

REVIEW BASIS. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subset X$
$\mathcal{B}$ **linear independent**: $c_1\vec{v}_1 + ... + c_n\vec{v}_n = 0$ implies $c_1 = ... = c_n = 0$.
$\mathcal{B}$ **span** X: $\vec{v} \in X$ then $\vec{v} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$.
$\mathcal{B}$ **basis**: both linear independent and span.



BASIS: ENOUGH BUT NOT TOO MUCH. The spanning condition for a basis assures that there are **enough** vectors to represent any other vector, the linear independence condition assures that there are **not too many** vectors. A basis is, where J.Lo meets A.Hi: Left: J.Lopez in **"Enough"**, right "The man who new **too much**" by A.Hitchcock

DIMENSION. The number of elements in a basis of X is independent of the choice of the basis. This works because if q vectors span X and p other vectors are independent then $q \geq p$ (see lemma) Applying this twice to two different basises with q or p elements shows $p = q$. The number of basis elements is called the **dimension** of X.

UNIQUE REPRESENTATION. $\vec{v}_1, \dots, \vec{v}_n \in X$ **basis** $\Rightarrow$ every $\vec{v} \in X$ can be written uniquely as a sum $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

EXAMPLES. The dimension of $\{0\}$ is zero. The dimension of any line 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line in space have dimension 1.

REVIEW: KERNEL AND IMAGE. We can construct a basis of the kernel and image of a linear transformation $T(x) = Ax$ by forming $B = \mathrm{rref} A$. The set of Pivot columns in A form a basis of the image of T, a basis for the kernel is obtained by solving $Bx = 0$ and introducing free variables for each non-pivot column.

PROBLEM. Find a basis of the span of the column vectors of A

$$A = \begin{bmatrix} 1 & 11 & 111 & 11 & 1 \\ 11 & 111 & 1111 & 111 & 11 \\ 111 & 1111 & 11111 & 1111 & 111 \end{bmatrix}.$$

Find also a basis of the **row space** the span of the row vectors.

SOLUTION. In order to find a basis of the column space, we row reduce the matrix A and identify the leading 1: we have

$$\mathrm{rref}(A) = \begin{bmatrix} \boxed{1} & 0 & -10 & 0 & 1 \\ 0 & \boxed{1} & 11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because the first two columns have leading $\boxed{1}$, the first two columns of A span the image of A, the column space. The basis is $\{ \begin{bmatrix} 1 \\ 11 \\ 111 \end{bmatrix}, \begin{bmatrix} 11 \\ 111 \\ 1111 \end{bmatrix} \}$.
Now produce a matrix B which contains the rows of A as columns

$$B = \begin{bmatrix} 1 & 11 & 111 \\ 11 & 111 & 1111 \\ 111 & 1111 & 11111 \\ 11 & 111 & 1111 \\ 1 & 11 & 111 \end{bmatrix}$$

and row reduce it to

$$\mathrm{rref}(B) = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first two columns of A span the image of B. $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 11 \\ 111 \\ 11 \\ 1 \end{bmatrix} \begin{bmatrix} 11 \\ 111 \\ 1111 \\ 111 \\ 11 \end{bmatrix} \}$.

Mathematicians call a fact a "lemma" if it is used to prove a theorem and if does not deserve the be honored by the name "theorem":

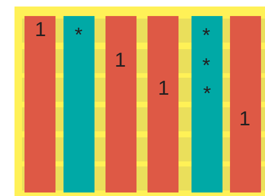
LEMMA. If q vectors $\vec{w}_1, ..., \vec{w}_q$ span X and $\vec{v}_1, ..., \vec{v}_p$ are linearly independent in X, then $q \geq p$.

REASON. Assume $q < p$. Because $\vec{w}_i$ span, each vector $\vec{v}_i$ can be written as $\sum_{j=1}^q a_{ij}\vec{w}_j = \vec{v}_i$. Now do Gauss-Jordan elimination of the augmented $(p \times (q+n))$-matrix to this system: $\begin{vmatrix} a_{11} & \dots & a_{1q} & | & \vec{w}_1^T \\ \dots & \dots & \dots & | & \dots \\ a_{p1} & \dots & a_{pq} & | & \vec{w}_q^T \end{vmatrix}$, where $\vec{w}^T$ is the vector $\vec{v}$ written as a row vector. Each row of A of this $\begin{bmatrix} A|b \end{bmatrix}$ contains some nonzero entry. We end up with a matrix, which contains a last row $\begin{vmatrix} 0 & ... & 0 & | & b_1\vec{w}_1^T + ... + b_q\vec{w}_q^T \end{vmatrix}$ showing that $b_1\vec{w}_1^T + \dots + b_q\vec{w}_q^T = 0$. Not all b_j are zero because we had to eliminate some nonzero entries in the last row of A. This nontrivial relation of $\vec{w}_i^T$ (and the same relation for column vectors $\vec{w}$) is a contradiction to the linear independence of the $\vec{w}_j$. The assumption $q < p$ can not be true.

THEOREM. Given a basis $\mathcal{A} = \{v_1, ..., v_n\}$ and a basis $\mathcal{B} = \{w_1, ..., .w_m\}$ of X, then $m = n$.

PROOF. Because $\mathcal{A}$ spans X and $\mathcal{B}$ is linearly independent, we know that $n \leq m$. Because $\mathcal{B}$ spans X and $\mathcal{A}$ is linearly independent also $m \leq n$ holds. Together, $n \leq m$ and $m \leq n$ implies $n = m$.

DIMENSION OF THE KERNEL. The number of columns in rref(A) without leading 1's is the **dimension of the kernel** dim(ker(A)): we can introduce a parameter for each such column when solving $Ax = 0$ using Gauss-Jordan elimination. The dimension of the kernel of A is the number of "free variables" of A.



DIMENSION OF THE IMAGE. The number of **leading 1** in rref(A), the rank of A is the **dimension of the image** dim(im(A)) because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.

RANK-NULLETETY THEOREM Let $A : \mathbf{R}^n \to \mathbf{R}^m$ be a linear map. Then

$$\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n$$

This result is sometimes also called the **fundamental theorem of linear algebra**.
SPECIAL CASE: If A is an invertible $n \times n$ matrix, then the dimension of the image is n and that the $\dim(\ker)(A) = 0$.

PROOF OF THE DIMENSION FORMULA. There are n columns. dim(ker(A)) is the number of columns without leading 1, dim(im(A)) is the number of columns with leading 1.

FRACTAL DIMENSION. Mathematicians study objects with non-integer dimension since the early 20'th century. The topic became fashion in the 80'ies, when mathematicians started to generate fractals on computers. To define fractals, the notion of dimension is extended: define a **s-volume of accuracy** r of a bounded set X in $\mathbf{R}^n$ as the infimum of all $h_{s,r}(X) = \sum_{U_j} |U_j|^s$, where U_j are cubes of length $\leq r$ covering X and $|U_j|$ is the length of U_j. The **s-volume** is then defined as the limit $h_s(X)$ of $h_s(X) = h_{s,r}(X)$ when $r \to 0$. The **dimension** is the limiting value s, where $h_s(X)$ jumps from 0 to ∞. Examples:
1) A smooth curve X of length 1 in the plane can be covered with n squares U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^n (1/n)^s = n(1/n)^s$. If $s < 1$, this converges, if $s > 1$ it diverges for $n \to \infty$. So $\dim(X) = 1$.
2) A square X in space of area 1 can be covered with n^2 cubes U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n^2} (1/n)^s = n^2(1/n)^s$ which converges to 0 for $s < 2$ and diverges for $s > 2$ so that $\dim(X) = 2$.

3) The **Shirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k'th step, we need 8^k squares of length $1/3^k$ to cover the carpet. The s−volume $h_{s,1/3^k}(X)$ of accuracy $1/3^k$ is $8^k(1/3^k)^s = 8^k/3^{ks}$, which goes to 0 for $k \to \infty$ if $3^{ks} < 8^k$ or $s < d = \log(8)/\log(3)$ and diverges if $s > d$. The dimension is $d = \log(8)/\log(3) = 1.893$..