



REMARK. The problem to find a basis for all vectors  $\vec{w}_i$  which are orthogonal to a given set of vectors, is equivalent to the problem to find a basis for the kernel of the matrix which has the vectors  $\vec{w}_i$  in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the  $m \times n$  matrix  $A$  into the form  $\text{rref}(A)$ . Call a column a **pivot column**, if it contains a leading 1. The corresponding set of column vectors of the original matrix  $A$  form a basis for the image because they are linearly independent and are in the image. Assume there are  $k$  of them. They span the image because there are  $(k - n)$  non-leading entries in the matrix.

REMARK. The problem to find a basis of the subspace generated by  $\vec{v}_1, \dots, \vec{v}_n$ , is the problem to find a basis for the image of the matrix  $A$  with column vectors  $\vec{v}_1, \dots, \vec{v}_n$ .

EXAMPLES.

- 1) Two vectors on a line are linear dependent. One is a multiple of the other.
- 2) Three vectors in the plane are linear dependent. One can find a relation  $a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$  by changing the size of the lengths of the vectors  $\vec{v}_1, \vec{v}_2$  until  $\vec{v}_3$  becomes the diagonal of the parallelogram spanned by  $\vec{v}_1, \vec{v}_2$ .
- 3) Four vectors in three dimensional space are linearly dependent. As in the plane one can change the length of the vectors to make  $\vec{v}_4$  a diagonal of the parallelepiped spanned by  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ .

EXAMPLE. Let  $A$  be the matrix  $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ . In reduced row echelon form is  $B = \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

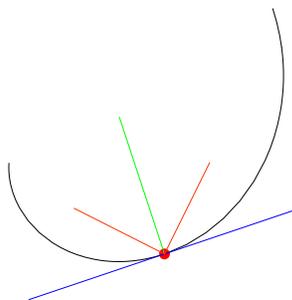
To determine a basis of the kernel we write  $Bx = 0$  as a system of linear equations:  $x + y = 0, z = 0$ . The variable  $y$  is the free variable. With  $y = t$ ,  $x = -t$  is fixed. The linear system  $\text{rref}(A)x = 0$  is solved by

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \text{ So, } \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ is a basis of the kernel.}$$

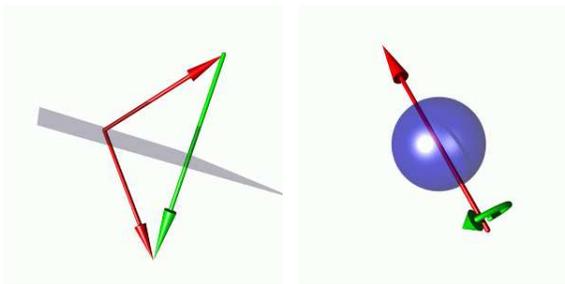
EXAMPLE. Because the first and third vectors in  $\text{rref}(A)$  are columns with leading 1's, the first and third

columns  $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  of  $A$  form a basis of the image of  $A$ .

WHY DO WE INTRODUCE BASIS VECTORS? Wouldn't it be just easier to look at the standard basis vectors  $\vec{e}_1, \dots, \vec{e}_n$  only? The reason for more general basis vectors is that they allow a **more flexible adaptation** to the situation. A person in Paris prefers a different set of basis vectors than a person in Boston. We will also see that in many applications, problems can be solved easier with the right basis.



For example, to describe the reflection of a ray at a plane or at a curve, it is preferable to use basis vectors which are tangent or orthogonal. When looking at a rotation, it is good to have one basis vector in the axis of rotation, the other two orthogonal to the axis. Choosing the right basis will be especially important when studying differential equations.



A PROBLEM. Let  $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ . Find a basis for  $\ker(A)$  and  $\text{im}(A)$ .

SOLUTION. From  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$  we see that  $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$  is in the kernel. The two column vectors

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, [2, 1, 1]$  of  $A$  form a basis of the image because the first and third column are pivot columns.