

Name:

MWF10 Izzet Coskun
MWF11 Oliver Knill

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total:		130

Problem 1) (20 points) True or False? No justifications are needed.

T F

If A is a symmetric matrix such that $A^5 = 0$, then $A = 0$.

Solution:

Diagonalize

T F

If A and B are 3×3 symmetric matrices, then AB is symmetric.

Solution:

$(AB)^T = B^T A^T = BA$ is not equal to AB since A and B do not need to commute.

T F

The differential equation $f'''(x) + f''(x) + f(x) = \sin(x)$ has a three dimensional solution set.

Solution:

The homogenous equation has a three dimensional solution set. The general solution is the sum of a special solution and a homogeneous solution.

T F

The initial value problem $f'''(x) + f''(x) + f(x) = \sin(x)$, $f(0) = 0$, $f'(0) = 0$ has exactly one solution.

Solution:

The solution space is one dimensional. Without any initial conditions, we have a three dimensional solution set. The initial condition $f(0) = 0$ fixes one dimension, the other initial condition an other one.

T F

Every real 3×3 matrix having $1 + i$ as an eigenvalue is diagonalizable over the complex numbers.

Solution:

Because the matrix A is real, $1 - i$ is an other eigenvalue. Because a polynomial of degree 3 has at least one real root, we have additionally a real eigenvalue. So, the matrix has 3 different eigenvalues and is therefore diagonalizable.

T F

If A is a nonzero diagonalizable 4×4 matrix, then A^4 is nonzero.

Solution:

Because A is diagonalizable and nonzero, it has at least one nonzero eigenvalue λ with eigenvector v . Because $A^n v = \lambda^n v \neq 0$, A^n can not be zero for all n . Especially not for $n = 4$.

T F

There exists a real 2×2 matrix A such that $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution:

Take a rotation-dilation matrix which is associated to the complex number $z = i$.

T F

There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det(A) + \det(B)$.

Solution:

For rotation dilation matrices $\det(A) = a^2 + b^2$, is the square of the length of the associated vector (a, b) respectively the complex number $z = a + ib$. Just take two orthogonal vectors. Pythagoras will assure that the addition determinant formula is true.

T F

The kernel of the linear map D^{100} on $C^\infty(\mathbf{R})$ has dimension 100.

Solution:

The kernel consists of all polynomials of degree 99.

T F

$Tf(x) = \sin(x)f(x) + f(0) + \int_{-1}^x f(y) dy$ is a linear map on $C^\infty(\mathbf{R})$.

Solution:

Check $T(f + g) = T(f) + T(g)$, $T(\lambda f) = \lambda T(f)$, $T(0) = 0$.

T F

If $S^{-1}AS = B$, then $\text{tr}(A)/\text{tr}(B) = \det(A)/\det(B)$.

Solution:

Indeed, for similar matrices, both the trace and the determinant agree, so that both sides are 1.

T F

If a 3×3 matrix A is invertible, then its rows form a basis.

Solution:

If A is invertible, so is A^T . The columns of A^T which are the rows of A form a basis.

T F

A 4×4 orthogonal matrix has always a real eigenvalue.

Solution:

Build a partitioned matrix which has two 2×2 rotation matrices in the diagonal and zero matrices in the side diagonal.

T	F
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If A is orthogonal and B satisfies $B^2 = 1$ then AB has determinant 1 or -1 .

Solution:

Both A and B have determinant 1 or -1 .

T	F
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If $\dot{x} = Ax$ is asymptotically stable, then $\dot{x} = -Ax$ is not asymptotically stable.

Solution:

The real part of the eigenvalues of $-A$ are positive.

T	F
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If $\dot{x} = Ax$ is asymptotically stable, then the differential equation $\dot{x} = Ax + (x \cdot x)x$ has a stable origin.

Solution:

The Jacobean matrix is A . It determines the asymptotic stability also in the nonlinear case.

T	F
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The map on $C^\infty(\mathbf{R})$ given by $T(f)(t) = t + f(t)$ is linear.

Solution:

The constant function 0 is not mapped into 0: $T(0)$ is the function $f(t) = t$.

T	F
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0 is a stable equilibrium for the discrete dynamical system $\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$.

Solution:

The trace is 2, the determinant is 2. This is not inside the stability triangle.

T	F
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If A is an arbitrary 4×4 matrix, then A and A^T are similar.

Solution:

It would be true for diagonalizable matrices. The shear is a counter example for 2×2 matrices.

T	F
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If A is an invertible 4×4 matrix, then the unique least squares solution to $Ax = b$ is $A^{-1}b$.

Solution:

Yes, in that case, the least squares solution is the real solution. One can also see it formally from $(A^T A)^{-1} A^T = A^{-1}$.

Total

Problem 2) (10 points)

Match the following objects with the correct description. Every equation matches exactly one description.

- a) $\dot{x} = 3x - 5y, \dot{y} = 2x - 3y$
- b) $f_t = f_{xx} + f_{yy}$.
- c) $D^2 f(x) + Df(x) - f(x) = \sin(x)$
- d) $\dot{x} = 3x^3 - 5y, \dot{y} = x^2 + y^2 + 2$
- e) $\dot{x} + 3x = 0$.

- i) An Inhomogenous linear ordinary differential equation
- ii) A partial differential equation
- iii) A linear two dimensional ordinary differential equation
- iv) A homogeneous one-dimensional first order linear ordinary differential equation.
- v) A nonlinear ordinary differential equation.

Solution:

- a) iii)
- b) ii)
- c) i)
- d) v)
- e) iv)

Problem 3) (10 points)

Define $A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix}$.

a) Find $\text{rref}(A)$, the reduced row echelon form of A .

b) Find a bases for $\ker(A)$ and $\text{im}(A)$.

c) Find an orthonormal basis for $\ker(A)$.

d) Verify that $\mathbf{v} \in \ker(A)$, where $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$.

e) Express \mathbf{v} in terms of your orthonormal basis for $\ker(A)$.

Solution:

a) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) The first two columns contain leading 1. Therefore, the first two columns $\begin{bmatrix} 1 \\ -5 \\ 9 \end{bmatrix}$ and

$\begin{bmatrix} -2 \\ 6 \\ -10 \end{bmatrix}$ of A form a basis for the image of A . To get a basis for the kernel, produce free variables s, t for the last two columns. The equations $z = s, w = t, y - 2s + 3t = 0, x - s + 2t = 0$ show that the general kernel element is $(s - 2t, 2s - 3t, s, t)$. Therefore

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the kernel.

c) Do Gram-Schmidt orthogonalization: $\vec{w}_1 = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \vec{v}_2 - (\vec{v}_2, \vec{w}_1)\vec{w}_1 = \vec{v}_2 -$

$4 \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ and normalize to get $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} / \sqrt{6}$.

d) Just check that $A\vec{v} = \vec{0}$.

e) $\vec{v} = (\vec{v} \cdot \vec{w}_1)\vec{w}_1 + (\vec{v} \cdot \vec{w}_2)\vec{w}_2 = -4\vec{w}_1 - 2\vec{w}_2$.

Problem 4) (10 points)

Find all solutions to the differential equation

$$f''(t) - 2f'(t) + f(t) = 4e^{3t}.$$

Find the unique solution given the initial conditions $f(0) = 1$ and $f'(0) = 1$.

Solution:

First find a solution to the homogeneous equation: $(D^2 - 2D + 1)f = 0$. Because $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, the homogeneous solution is a linear combination of e^t and te^t . $f(t) = ae^t + bte^t$.

To get a special solution, we try $f = ce^{3t}$. Plugging this into the differential equation gives $(9c - 6c + c)e^{3t} = 4e^{3t}$ so that $c = 1$. The general solution is $f(t) = ae^t + bte^t + e^{3t}$. The conditions $f(0) = 1$ and $f'(0) = 1$ fixes the constants: $a + 1 = 1, a + b + 3 = 1$: $a = 0$ and $b = -2$. The unique solution is $f(t) = -2te^t + e^{3t}$.

Problem 5) (10 points)

- a) Write $f(x) = |\sin(x)|$ on $[-\pi, \pi]$ as a Fourier series for f .
- b) Find the solution to the heat equation $T_t = \mu T_{xx}$ with $T(x, 0) = f(x)$.
- c) Find the solution to the wave equation $T_{tt} = c^2 T_{xx}$ with $T(x, 0) = f(x)$ and for which $T_t(x, 0) = 0$ holds for all x .

Solution:

a) Since f is even, we have only to compute the cos series $f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$.

$$\begin{aligned} a_n &= 2/\pi \int_0^\pi \cos(nx) \sin(x) dx \\ &= (2/\pi)(1/2) \int_0^\pi \sin((n+1)x) - \sin((n-1)x) dx \end{aligned}$$

which is $\frac{1}{\pi}(\frac{2}{n+1} - \frac{2}{n-1}) = \frac{4}{\pi(1-n^2)}$ if n is even and 0 if n is odd.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin(x)|/\sqrt{2} dx \\ &= \frac{2}{\sqrt{2}\pi} \int_0^\pi \sin(x) dx = \frac{4}{(\sqrt{2}\pi)}. \end{aligned}$$

Therefore,

$$f(x) = \frac{4}{(\sqrt{2}\pi)} \frac{1}{\sqrt{2}} + \sum_{n \text{ even}} \frac{4}{\pi(1-n^2)} \cos(nx).$$

b) $T(x, t) = \frac{4}{(\sqrt{2}\pi)} \frac{1}{\sqrt{2}} + \sum_{n \text{ even}} \frac{4}{\pi(1-n^2)} e^{-\mu n^2 t} \cos(nx)$.

c) $T(x, t) = \frac{4}{(\sqrt{2}\pi)} \frac{1}{\sqrt{2}} + \sum_{n \text{ even}} \frac{4}{\pi(1-n^2)} \cos(nct) \cos(nx)$.

Problem 6) (10 points)

Find a single 3×3 matrix A for which all of the following properties are true.

a) The kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for A .

c) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the image of A .

Solution:

Because of c), we can put the third vector as a column vector. The puzzle is to complete

$A = \begin{bmatrix} 1 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ -1 & \cdot & \cdot \end{bmatrix}$, so that b) and a) hold. Because of b), we know that the third column is

equal to the negative of the first column. We are left with the puzzle $A = \begin{bmatrix} 1 & \cdot & -1 \\ 2 & \cdot & -2 \\ -1 & \cdot & 1 \end{bmatrix}$

knowing that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the kernel. But that determines also the entries in the middle

column $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

Problem 7) (10 points)

- a) Find all solutions to the differential equation $(D^2 - 3D + 2)f = 60e^{7x}$.
- b) Find all solutions to the differential equation $(D^2 - 2D + 1)f = x$.
- c) Find all solutions to the differential equation $(D^2 + 1)f = x^2$.

Solution:

a) We have $(D^2 - 3D + 2) = (D - 1)(D - 2)$. The general homogeneous solution is $ae^x + be^{2x}$. A special solution is obtained by trying ce^{7x} which gives $(49c - 21c + 2c)e^{7x} = 60e^{7x}$ or $c = 2$. The general solution is $\boxed{2e^{7x} + ae^x + be^{2x}}$.

b) We have $(D^2 - 2D + 1) = (D - 1)^2$. The general homogeneous solution is $ae^x + bxe^x$. A special solution is obtained by trying $f(x) = ax + b$ for which $(D^2 - 2D + 1)f = -2a + (ax + b) = ax + (-2a + b)$ such that $a = 1, b = 2$. The general solution is of the form $\boxed{2 + x + ae^x + bxe^x}$.

c) The general homogeneous solution is $a \cos(x) + b \sin(x)$. A special inhomogeneous solution is obtained by trying $f(x) = ax^2 + bx + c$ for which $(D^2 + 1)f = 2a + ax^2 + bx + c$ in order that this is x^2 , we have $a = 1, c = -2, b = 0$. The general solution is of the form $\boxed{x^2 - 2 + a \cos(x) + b \sin(x)}$.

Problem 8) (10 points)

Find the matrix for the rotation in \mathbf{R}^3 by 90° about the line spanned by $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, in a clockwise direction as viewed when facing the origin from the point $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. You get full credit if you leave the result written as a product of matrices or their inverses.

Solution:

We find an orthonormal basis \mathcal{B} for which one vector is in the line: $[2, 2, 1]/3, [2, -2, 0]/\sqrt{8}, [2, 2, -8]/\sqrt{72}$ will do. Form the matrix $S =$

$\begin{bmatrix} 2/3 & 2/\sqrt{8} & 2/\sqrt{72} \\ 2/3 & -2/\sqrt{8} & 2/\sqrt{72} \\ 1/3 & -8/\sqrt{8} & -8/\sqrt{72} \end{bmatrix}$ which brings you from the standard basis to the adapted

basis \mathcal{B} . The matrix S^{-1} transforms from the new basis to the standard basis. In the standard basis, a rotation around the x axes in the clockwise direction when facing the

origin is given by $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. The matrix, we are looking for is $A = SBS^{-1}$.

Problem 9) (10 points)

a) Find the eigenvalues of the matrix $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

b) Is $\vec{0}$ a stable equilibrium point for the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} ?$$

c) Describe, how the solution curves of $\frac{d\vec{x}}{dt} = A\vec{x}$ look like.

d) Is $\vec{0}$ a stable equilibrium for the discrete dynamical system $x_{n+1} = Ax_n$?

Solution:

a) The eigenvalues are $(1 + i)/2, (1 - i)/2$.

b) The real part of the eigenvalues is positive. The system is unstable.

c) Except for the initial point 0, all solutions spiral out. When following a solution, we escape to infinity.

d) Yes, the eigenvalues have norm < 1 . The solutions are stable.

Problem 10) (10 points)

Does the system

$$\frac{d\vec{x}}{dt} = B\vec{x}$$

with

$$B = \begin{bmatrix} 0 & -1 & -9 & -9 & -8 \\ 0 & 0 & 0 & -1 & -9 \\ 5 & 0 & 5 & 0 & -5 \\ 1 & 9 & 0 & 0 & 0 \\ 1 & 9 & 9 & 8 & 0 \end{bmatrix}$$

have a stable origin?

Hint No lengthy computations are needed. Especially, no eigenvalues have to be computed. If $\lambda_1, \dots, \lambda_5$ are the eigenvalues, can you say something about their sum?

Solution:

If the system were stable, then all eigenvalues had a negative real part. Also the sum had a negative real part. The sum is however equal to 5 because it is the trace of B .

Problem 11) (10 points)

A 4×4 matrix A is called **symplectic** if $AJA^T = J$, where $J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

- a) Verify that J itself is symplectic.
- b) Show that if A is symplectic, then A is invertible and A^{-1} is symplectic.
- c) Check that if both A and B are symplectic, then AB is symplectic.
- d) For $v, w \in \mathbf{R}^4$, define $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T J \mathbf{w}$. Is this an inner product on \mathbf{R}^4 ? Why or why not?
- e) Show that for a symplectic matrix A , one has $\det(A) = 1$ or $\det(A) = -1$.

Solution:

- a) Follows from $JJ^T = 1$.
- b) If A were not invertible, then AJA^T were not invertible. But J is invertible. $AJA^T = J$ is equivalent to $A^{-1}J(A^{-1})^T = J$.
- c) $ABJ(AB)^T = ABJB^T A^T = AJA^T = J$.
- d) This is not an inner product. From $(ABC)^T = C^T B^T A^T$ for general matrices, we know that $\langle \mathbf{w}, \mathbf{v} \rangle = -\langle \mathbf{v}, \mathbf{w} \rangle$. Especially, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.
- e) From $\det(A) = \det(A^T)$ and the product formula for the determinant and since $\det(J) = 1$, we have $\det(A)^2 = 1$.

Remark. One can actually show that symplectic matrices always satisfy $\det(A) = 1$, but this is more difficult to establish.

Problem 12) (10 points)

Find the ellipse $f(x, y) = ax^2 + by^2 - 1 = 0$ which best fits the data $(2, 2), (-1, 1), (-1, -1), (2, -1)$.

Solution:

We write down a linear system of equations for the unknowns (a, b) which tells that all the data points are on the ellipse. The least square solution to this system is our best guess.

$$\begin{aligned} 4a + 4b &= 1 \\ a + b &= 1 \\ a + b &= 1 \\ 4a + b &= 1 \end{aligned}$$

which can be written as $Ax = b$ with $b = [1, 1, 1, 1]^T$ and $A = \begin{bmatrix} 4 & 4 \\ 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{bmatrix}$. Calculate

$$A^T A = \begin{bmatrix} 34 & 22 \\ 22 & 19 \end{bmatrix} \text{ and } A^T b = \begin{bmatrix} 10 \\ 7 \end{bmatrix} \text{ and from that } \begin{bmatrix} a \\ b \end{bmatrix} = (A^T A)^{-1} A^T b = \begin{bmatrix} 2/9 \\ 1/9 \end{bmatrix}.$$

The best ellipse is $2x^2 + y^2 = 9$.