Name:

MWF10 Izzet Coskun

MWF11 Oliver Knill

- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points) True or False? No justifications are needed.

T	F	All symmetric real matrices are diagonalizable.

T	**F**	There exists a 3×3 real symmetric matrix whose Jordan-normal form is $\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

T	F	If A is any matrix, then both AA^T and $A^T A$ are orthogonally diagonalizable.

T	F	All orthogonal projections are diagonalizable.

T	F	If the regression line $y = ax + b$ obtained by fitting some data $\{(x_1, y_1), ..., (x_m, y_m)\}$ happens to contain all datapoints, then the corresponding least square solution of $A\vec{x} = \vec{b}$ is an actual solution of $A\vec{x} = \vec{b}$.

T	F	$\det\left(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}\right) = -7.$

T	**F**	There exists a symmetric 2×2 matrix A such that $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

T	F	The kernel of the operator $(D-2)^5$ is spanned by e^{2t}, te^{2t}, $t^2 e^{2t}$, $t^3 e^{2t}$, $t^4 e^{2t}$.

T	F	Let A be a 2×2 matrix. The system $\frac{dx}{dt} = Ax$ is asymptotically stable if and only if the eigenvalues of A have negative real parts.

T	**F**	Let A be a 2×2 matrix. The discrete dynamical system $A(t+1) = Ax(t)$ is asymptotically stable if and only if the eigenvalues of A have negative real parts.

T	**F**	The subset of $X = C^\infty(\mathbf{R})$, the set of smooth functions of the real line, defined by $Y = \{f \in C^\infty(\mathbf{R}) : f(0) = 1\}$ is a linear subspace of X.

T	F	The subset of $C^\infty(\mathbf{R})$ defined by $Y = \{f \in C^\infty(\mathbf{R}) : f(0) = f''(2)\}$ is a linear subspace of X.

T	F	The operator $T(f) = (D^2 + 12tD + 17)f$ defines a linear map from $C^\infty(\mathbf{R})$ to $C^\infty(\mathbf{R})$.

T	F	The operator $T(f) = (D^2 + 12t^3 D + 17t^2)f$ defines a linear map from $C^\infty(\mathbf{R})$ to $C^\infty(\mathbf{R})$.

T	**F**	If A is 2×2 matrix with $\det(A) < 0$, then the system $\frac{dx}{dt} = Ax$ has 0 as a stable equilibrium.

T	F	In the Fourier series expansion of the function $t+1$ on $[-\pi, \pi]$, the coefficients a_n belonging to $\cos(nt)$ are zero for all $n \geq 1$.

T	**F**	If a 2×2 matrix A has the eigenvalues $-2, -1$, then the orbits of system $x(t) \mapsto x(t+1) = Ax(t)$ stay bounded.

T	F	If a 2×2 matrix A has an eigenvalues $-2, -1$, then the orbits of the system $\frac{d}{dt}x(t) = Ax(t)$ stay bounded.

Problem 2) (10 points)

Match the following differential equations with the correct description. Every equation matches exactly one description. No justifications are necessary.

a)
$$\dot{x} = 3x - 5y$$
$$\dot{y} = 2x - 3y$$

b)
$$\dot{x} = -4y + 2x^2 + 2x^3$$
$$\dot{y} = 4y(1 - x^2)$$

c)
$$\dot{x} = -x + 2y - y^2$$
$$\dot{y} = 3x - y - xy - y^2$$

d)
$$\dot{x} = 3x - 5y$$
$$\dot{y} = x^2 + y^2 + 2$$

e)
$$\dot{x} = 2y(x - y) - x$$
$$\dot{y} = y(x - y) - y$$

Fill in 1),...,5) here.

a)	b)	c)	d)	e)

1) The equation has a stable equilibrium at $x = 1, y = 1$.
2) The equation has an unstable equilibrium at $x = 1, y = 1$.
3) The equation has a non-constant solution which stays on the line $x = y$.
4) The equation has a closed periodic orbit.
5) The equation has no equilibria.

Solution:
1) matches with c). The Jacobian matrix at $(1, 1)$ has the eigenvalues $-1, -3$.
2) matches with b). The Jacobian matrix at $(1, 1)$ has the trace 10 and can not be stable.
3) matches with e). If $x = y$, then $\dot{x} = -x; \dot{y} = -y$.
4) matches with a). The solution curves are on ellipses.
5) matches with d). Note that $\dot{y}$ is never 0

a)	b)	c)	d)	e)
4	2	1	5	3

Problem 3) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix}$.

a) Find a basis for the kernel of A.

b) Find a basis for the image of A.

Solution:

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ b) The first, second and forth column contain leading

ones. Therefore, these columns form a basis for the image of A $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$

a) Introduce free variables s, t for the third and fifth row and solve the system $\text{rref} A \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix}.$

This shows $u = t, y = 0, z = s, x = -s - t, v = -t.$ So $[x, y, z, u, v]^T = [-s + t, 0, s, t, -t]^T$

so that $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ is a basis for the kernel.

Problem 4) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$

a) Find all possibly complex eigenvalues of A with their algebraic multiplicities.
b) Does A have a possibly complex eigenbasis? If so, find one.
c) Is A diagonalizable? Why or why not?
d) Let T be the linear transformation defined by $T(v) = Av$. Describe T geometrically.

Solution:
a) The characteristic polynomial is $(\lambda - 1)^2(\lambda + 1)^2$. The eigenvalues are $1, -1$ with algebraic multiplicities 2.

b) Yes, there is an eigenbasis: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$

c) Yes, it is diagonalizable. We have a complete set of eigenvectors d) It permutes the standard basis vectors. It is a reflection in four dimensional space.

Find the function $f(x) = a + b\cos(x)$ which best fits the data

$$\begin{aligned}(x_1, y_1) &= (0, 1) \\ (x_2, y_2) &= (\pi/2, -1) \\ (x_3, y_3) &= (\pi, 1) \\ (x_4, y_4) &= (2\pi, 1)\end{aligned}$$

Solution:
Setting up the equations $a + b\cos(x_i) = y_i$ gives the system

$$\begin{aligned}a + b &= 1 \\ a &= -1 \\ a - b &= 1 \\ a + b &= 1\end{aligned}$$

which can be written as $A\vec{x} = \vec{b}$, where $\vec{x} = [a, b]^T$ and $\vec{b} = [1, -1, 1, 1]^T$. The least square solution is given by $(A^T A)^{-1} A^T \vec{b}$. One has $A^T A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ and $A^T \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $(A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 5/11 \\ 2/11 \end{bmatrix}$. The best fit is given by the function $\boxed{f(t) = 5/11 + 2/11\cos(t)}$.

a) Find the solution of the differential equation $f'(t) + 3f(t) = e^{-2t}$, $f(0) = 0$.
b) Find the general solution of $f''(t) + 4f'(t) + 3f(t) = 1$.
with $f(0) = 1/3$, $f(1) = 1/3 + 1/e^3 - 1/e$.
c) Find the solution of $f''(t) = -4f(t)$ with $f(0) = 1$, $f'(0) = 2$.

Solution:
a) The general solution of the homogeneous equation $(D+3)f = 0$ is e^{-3t}. For a special solution, try $f = e^{-2t}$ which gives $(D+3)f = -2 + 3f = f$. The general solution is therefore $f(t) = e^{-2t} + c3^{-3t}$. For $t = 0$, this is $1 + c$ such that $c = -1$.

$\boxed{\text{The solution is } f(t) = e^{-2t} - e^{-3t}.}$

b) The general solution of the homogeneous equation $(D^2+4D+3)f = (D+3)(D+1)f$ is $ae^{-3t} + be^{-t}$. A special solution of the inhomogeneous equation if $f(t) = 1/3$. Therefore, the general solution is $1/3 + ae^{-3t} + be^{-t}$. $\boxed{\text{The solution is } f(t) = 1/3 + e^{-3t} - e^{-t}.}$

c) The equation has the solutions $f(0)\cos(2t) + \frac{f'(0)}{2}\sin(2t) = \cos(2t) + \sin(2t)$.

Problem 7) (10 points)

a) (7 points) Find a 4×4 matrix A with entries $0, +1$ and -1 for which the determinant is maximal.

b) (3 points) Find the QR decomposition of A.

Solution:
a) This was a homework problem. The determinant is the volume of a parallelepiped. If its lengths are fixed, then it is maximal, if all sides are orthogonal. This can be achieved

with $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$

b) $A = Q \cdot R = (A/2) \cdot 2I_4$.

Problem 8) (10 points)

Define $f = \sinh(x) = \frac{e^x - e^{-x}}{2}$ on $C^\infty([-\pi, \pi])$ be a function on the interval $[-\pi, \pi]$. Find a solution $T(t, x)$ of the heat equation $\dot{T} = T_{xx}$ which satisfies $T(0, x) = f(x)$.

Hint. $\int \sinh(x)\sin(nx)\, dx = \frac{\cosh(x)\sin(nx) - n\cos(nx)\sinh(x)}{1+n^2}$. You can leave terms like $\sinh(\pi)$.

Solution:
$\sinh(x)$ is an odd function. So the Fourier coefficients are $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh(x)\sin(x)\, dx = \frac{2}{\pi}\int_0^\pi \sinh(x)\sin(nx)\, dx = \frac{2n\sinh(\pi)(-1)^{n+1}}{\pi(1+n^2)}$. The solution of the wave equation can now be written down:

$f(t, x) = \sum_{n=1}^\infty a_n e^{-n^2 t}\sin(nx)$ which is $\boxed{\sum_{n=1}^\infty \frac{2n\sinh(\pi)(-1)^{n+1}}{\pi(1+n^2)}e^{-n^2 t}\sin(nx)}$

a) Find the Fourier series of $|\sin(x/2)|$ on $C([-\pi, \pi])$.

Hint. $\int \sin(x/2) \cos(nx) \, dx = \frac{-\cos((\frac{1}{2}+n)x)}{2n+1} + \frac{\cos((\frac{1}{2}-n)x)}{2n-1}$.

b) Find $\sum_{n=1}^{\infty} (-1)^n \frac{1}{4n^2-1}$.

Hint. Evaluate $f(x)$ at π.

Solution:
a) The function is even so that it has a cos-Fourier series

$$f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \ .$$

We get

$$a_0 = \frac{2}{pi} \int_0^{\pi} \sin(x/2) \frac{1}{\sqrt{2}} \, dx = \frac{2}{\sqrt{2\pi}}$$

and

$$
\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^{\pi} \sin(x/2) \cos(nx) \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \sin(x/2 + nx) - \sin(-x/2 + nx) \, dx \\
&= \frac{1}{\pi} [-\cos(x/2 - nx)/(1/2 - n) + \cos(x/2 + nx)/(1/2 + n)|_0^{\pi}] \\
&= \frac{2}{\pi} [-\cos(\pi/2 - n\pi)/(1 - 2n) + \cos(\pi/2 + n\pi)/(1 + 2n)] \\
&= \frac{2}{\pi} [1/(1 - 2n) - 1/(1 + 2n)] \\
&= \frac{4}{\pi} \frac{1}{1 - 4n^2}
\end{aligned}
$$

$\boxed{a_n = \frac{4}{\pi} \frac{1}{1-4n^2}}$.

b) Evaluating f at $x = \pi$ gives $f(\pi) = 1$ and because $\cos(n\pi) = (-1)^{n+1}$ we have

$$1 = f(\pi) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{4n^2 - 1} \ .$$

The solution is $(1 - 2/\pi)\pi/4$ which is equal to $\boxed{(2 - \pi)/2 = -0.285398.}$

An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x - y + 1) \\ y(x + y - 3) \end{bmatrix}.$$

a) Find all equilibrium points and nullclines of this system in $x \geq 0, y \geq 0$.
b) Sketch the vector field of this system in the first quadrant $x \geq 0, y \geq 0$ indicating the direction of the vector field along the nullclines and inside the regions determined by the nullclines.
c) Are there any stable equilibrium points? Justify your answers.
d) If both species start with positive populations, can either become extinct? Explain.

Solution:
a) The nullclines are $x = 0, y = 0, x - y + 1 = 0$ and $x + y - 3 = 0$. The equilibrium points are $(0,0), (-1,0), (0,3)$ and $(1,2)$ but only $(0,0), (0,3), (1,2)$ are in the right quadrant.
b) On the x-axes $y = 0$, the vector field is $(x^2 + x, 0)$ which is positive on the positive axes. On the y-axes, the field is $y^2 - 3y$ which is pointing up for $y > 3$ and pointing down for $y < 3$. The field spirals out at $(1,2)$.

c) The Jacobean matrix is $\begin{bmatrix} 2x - y + 1 & -x \\ y & 2y - 3 + x \end{bmatrix}$. At $(1,2)$ this is $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ which

has eigenvalues $(3 \pm i\sqrt{7})/2$. $(0,3)$, the Jacobean is $\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}$. There is no asymptotic

stability for any equilibrium point.
d) Yes, there is one curve which emerges from the equilibrium point $(1,2)$ and which enters the equilibrum point $(0,3)$. If we start on that curve but not on the equilibrium point $(1,2)$, the solution with run into the equilibrium point $(0,3)$. On this curve only, it is possible that the first species dies out.
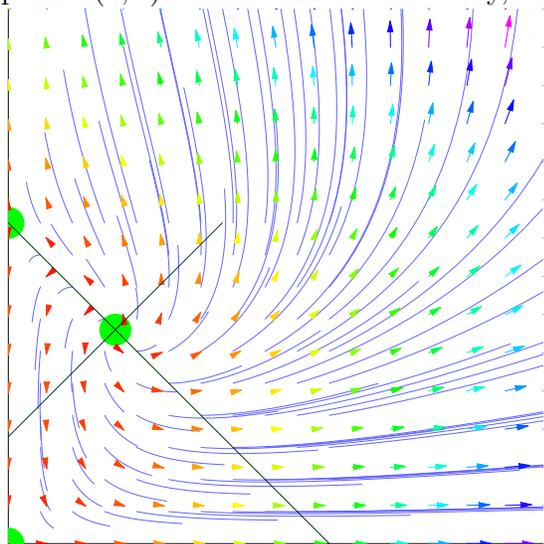
Problem 11) (10 points)

Consider the linear differential equation

$$\dot{x} = ax + y$$

$$\dot{y} = ay$$
$$\dot{z} = -z \, .$$

a) Write the system in the form $\frac{d}{dt}\vec{x} = A\vec{x}$, where A is a matrix.
b) For which parameters a is the system stable?

Solution:
a) The matrix is

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{bmatrix} \, .$$

b) In order that the system is stable, the eigenvalues of the matrix have to be all negative. This is equivalent to the fact that all the eigenvalues of $B = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ have a negative real part. And this is true if $\boxed{a < 0}$.