Last Name:________________________

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Mathematics 21b

Final Exam
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Your Section (circle one):

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MWF9	MWF 10	MWF 10	MWF 11	MWF 11	MWF 12	TuTh 10	TuTh 11:30

Question	Points	Score
1	14	
2	10	
3	8	
4	10	
5	8	
6	10	
7	10	
8	6	
9	10	
10	8	
11	6	
Total	100	

No calculators are allowed.
Justify your answers carefully (except for question 1).
Except for question 1, no credit can be given for unsubstantiated answers.

1. For each of the following, circle T for true or F for false. No explanation is necessary.

(a) T F If A is a real 4×4 matrix with two real eigenvalues, 3 and 4, with geometric multiplicities 1 and 3 respectively, then there are exactly 2 different diagonal matrices D for which there is an invertible S satisfying $S^{-1}AS = D$.

(b) T F $F(x,t) = e^{\mu n^2 t} e^{nx}$ is a solution of $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ for each integer n.

(c) T F If A is the matrix

$$\begin{bmatrix} -3 & 2 & 5 \\ 0 & a & -6 \\ 0 & 0 & -5 \end{bmatrix}$$

then the only values of the constant a for which the zero vector is an asymptotically stable equilibrium of the system $\vec{x}(t+1) = A\vec{x}(t)$ are $a < 0$.

(d) T F Every $n \times n$ matrix is diagonalizable over $\mathbf{C}$.

(e) T F The set of functions f in C^∞ for which $f(9) = 0$ is a subspace of C^∞.

(f) T F If A is a symmetric real 5×5 matrix with characteristic polynomial

$$f_A(\lambda) = \lambda^5 - 4\lambda^4 + 7$$

then A cannot be positive definite.

(g) T F If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ is a basis for $\mathbf{R}^n$ and $\vec{w_1}, \vec{w_2}, \ldots, \vec{w_n}$ is the result of applying Gram-Schmidt orthogonalization to the basis $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$, then the determinant of the matrix

$$\begin{bmatrix} | & | & & | \\ \vec{w_1} & \vec{w_2} & \cdots & \vec{w_n} \\ | & | & & | \end{bmatrix}$$

must be 1.

2. Let A be a 2×2 matrix such that $\mathrm{tr}(A) = 2$ and $\det(A) = 0$. Let λ_1, λ_2 be the eigenvalues of A. Suppose that $\lambda_1 > \lambda_2$.

a) Find λ_1 and λ_2.

b) If

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

find A.

c) Solve the continuous dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Your answer should be a closed formula for $\vec{x}(t)$.

d) Sketch the phase portrait of the system

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

e) Is the zero state an asymptotically stable equilibrium of this dynamical system ?

3. a) Find the Fourier series for $\cos(\frac{t}{2})$.
[Hint: You may find the formula $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$ useful]

b) Find a closed formula for

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

4. Let V be an m-dimensional subspace of $\mathbf{R^n}$ with an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_m$.
Let $\vec{v}_{m+1}, \ldots, \vec{v}_n$ be an orthonormal basis of $V^\perp$.
Then $\vec{v}_1, \ldots, \vec{v}_n$ is an orthonormal basis of $\mathbf{R^n}$.

Let R be the linear transformation given by reflection in the subspace V.

a) Let B be the matrix of R with respect to basis $\vec{v}_1, \ldots, \vec{v}_n$. Find B.

b) Let A be the matrix of R with respect to the standard basis. Find the eigenvalues of A and their geometric and algebraic multiplicities.

c) For $n = 3$, $m = 2$, and

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix},$$

find A.

5. Let A be an invertible $n \times n$ matrix and $\text{adj}(A)$ its classical adjoint matrix. Then A^{-1} is related to $\text{adj}(A)$ via

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

a) What is the relationship between $\det(\text{adj}(A))$ and $\det(A)$?

b) You are told that

$$\text{adj(A)} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 12 \\ 2 & 5 & -1 \end{bmatrix}$$

and that $\det A > 0$. Find A. [Hint: use part a) and the formula given above]

6. Let A be the matrix of the linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ given by reflection in the plane $x = y$. Let B be the matrix of the linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ given by reflection in the plane $x = z$.

a) Write down the matrices A and B.

b) Find all eigenvalues and eigenvectors of AB (including complex ones).

c) It is a fact that the linear transformation whose matrix is AB is a rotation. Find the axis of rotation and the angle of rotation.

7. For each of the following, give one example of a linear transformation $T : C^\infty \to C^\infty$ with that property.

a) Every real number is an eigenvalue of T.

b) 0 is an eigenvalue of T and the corresponding eigenspace E_0 is 3-dimensional.

c) Every non-zero function in C^∞ is an eigenfunction of T.

d) The kernel of T is the subspace P_2 consisting of all polynomials of degree 2 or less.

e) The image of T is 1-dimensional.

8. Find the least squares solution of the inconsistent linear system

$$\begin{aligned} x + 2y &= 1 \\ 2x - y &= 2 \\ x + 3y &= 4 \end{aligned}$$

9. a) Find all solutions to

$$\frac{d^3 f}{dx^3} - 2\frac{d^2 f}{dx^2} + 5\frac{df}{dx} = 0.$$

b) Find a simple polynomial solution to

$$\frac{d^3 f}{dx^3} - 2\frac{d^2 f}{dx^2} + 5\frac{df}{dx} = 10x.$$

c) Find all solutions to

$$\frac{d^3 f}{dx^3} - 2\frac{d^2 f}{dx^2} + 5\frac{df}{dx} = 10x.$$

10. Let V be the three-dimensional subspace of $\mathbf{R}^4$ spanned by the vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -4 \end{bmatrix}.$$

a) Find an orthonormal basis for V.

b) Find *all* vectors $\vec{w}$ in V for which $\vec{w}$ is orthogonal to both $\vec{v_1}$ and $\vec{v_2}$ and for which $||\vec{w}|| = 10$.

11. a) Find the value of k such that $f(t) = t + k$ and $g(t) = t^2$ are orthogonal in $C[-\pi, \pi]$.

b) Find the value of k such that $f(t) = t^2 + k$ and $g(t) = t^2$ are orthogonal in $C[-\pi, \pi]$.