Math 21b Final Exam Solutions - Spring 2001

1. True or False (no explanation is necessary):

- TRUE - Rotations and dilations commute.
- FALSE - From problem 5 of the second midterm, we know that A and B both have rank 3. So they each have kernels of dimension 2. Thus the 5x5 matrix BA can have a kernel of dimension at most 4. The zero matrix has kernel of dimension 5, so BA cannot be the zero matrix.
- TRUE - Recall from problem 5 of the first midterm that if A represents a shear $T : \mathbb{R}^2 \to \mathbb{R}^2$, then $(A - I_2)^2 = 0$. So any shear has characteristic polynomial $f_A(\lambda) = (\lambda - 1)^2 = \lambda^2 - 2\lambda + 1$. Thus the determinant of A must be 1.
- TRUE - Suppose A is a symmetric matrix. Represent a vector in the image of A by $A\vec{x}$. Suppose $\vec{y} \in \ker(A)$. If the kernel is the orthogonal complement of the image, then $(A\vec{x}) \cdot \vec{y}$ should be 0 regardless of which vectors $\vec{x}$ and $\vec{y}$ we chose. Recall that $A^T = A$ and observe:

  $$(A\vec{x}) \cdot \vec{y} = (A\vec{x})^T \vec{y} = (\vec{x}^T A^T)\vec{y} = (\vec{x}^T A)\vec{y} = \vec{x}^T (A\vec{y}) = \vec{x}^T \vec{0} = \vec{0}.$$

  Therefore the kernel and the image are orthogonal complements of each other.
- TRUE - Notice that AA^T is a symmetric matrix, and recall that the spectral theorem says that a symmetric matrix is orthogonally diagonalizable.
- FALSE - Consider the following matrices:

  $$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

  $ABA^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Notice that $\vec{v} = \langle 1, 0 \rangle$ is an eigenvector of B with eigenvalue 1, but $ABA^{-1}\vec{v} = \langle 1, -1 \rangle$, so $\vec{v}$ is not an eigenvector of ABA^{-1}.
- TRUE.
- FALSE - If any of the eigenvalues has nonnegative real part (either zero or positive), then the zero state is not a stable equilibrium.
- FALSE - Let $T : C^\infty \to C^\infty$ by the derivative transformation (linear). Then T maps every constant to 0, so T is not invertible.
- FALSE - Let $T : C^\infty \to C^\infty$ be the transformation defined by $T(f(x)) = xf(x)$. Notice that the kernel of T is $\{0\}$. However, none of the nonzero constants are in the image of T, so T is not invertible.

2. Find the determinant of each of the matrices and linear transformations (no explanation is necessary):

   - The are only two nonzero patterns in this matrix, both equal to 3. After counting the inversions, we see that they cancel, so the matrix has determinant 0.
   - Subtracting the first row from each of the other rows yields a matrix with only one nonzero pattern, having the same determinant as the given matrix. This pattern yields a product of 1, and it has an odd number of inversions, so the matrix has determinant -1.
   - There is only one nonzero pattern, and counting carefully, we see that it has 10 inversions. So the matrix has determinant 1.
   - Any vector in the plane is fixed, so there are two linearly independent vectors (in the plane) each with eigenvalue 1. Also, a vector normal to the plane is reflected to its negative, so $\langle 1, 2, 3\rangle$ is an eigenvector with eigenvalue -1. The product of the eigenvalues is the determinant of the matrix for this transformation, so the determinant is -1.
   - Since every multiple of the vector $\langle 1, 2, 3\rangle$ is projected onto the origin, the transformation is not invertible, so its matrix has determinant 0.

3. Match the graphs with the differential equations (no explanation is necessary):

   (a) III - A solution to the equation is $f(t) = e^{-t} + e^{2t}$, and this matches graph III.
   (b) I - A solution to the equation is $f(t) = e^{-t} + e^{t}$, and this matches graph I.
   (c) IV - A solution to the equation is $f(t) = k\sin(t)$, and this matches graph IV.
   (d) II - A solution to the equation is $f(t) = e^{t/4}(\sin t) + \sin(2t)$, and this matches graph II.
   (e) V - A solution to the equation is $f(t) = t\sin(t)$, and this matches graph V.

4. (a) $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$

   (b) $RS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; ST = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}; RST = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$

   (c) RS is reflection in the plane $y = z$. ST is rotation by 120° about the line $x = y = -z$ (in a clockwise direction when facing the origin from $\langle 1, 1, -1\rangle$). RST is rotation of 180° about the line $y = z = 0$. In other words, it is the reflection at the x axes.

5. (a) $E = \begin{bmatrix} 2 & 2 & 2 & 0 & 6 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 2 & 8 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}.$

   Performing elementary row operations gives:

$$\operatorname{rref}(E) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1/4 & -1/2 \\ 0 & 0 & 1 & 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1/2 & -1 \end{bmatrix}.$$

(b) Since the leading ones in the first five columns of E (referring to the matrix B) are in the first, third, and fourth columns, a basis for the image of B consists of the first, third, and fourth columns of B: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Also, from rref(E) we see that a basis for the kernel of B consists of: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -3 \\ 1 \end{bmatrix} \right\}$.

(c) Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(d) Let $C = D$ and let A be obtained from the three upper rows of the augmented part of $rref(E)$: $A = \begin{bmatrix} 0 & 0 & 1/4 & -1/2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1/4 & 1/2 \end{bmatrix}$.

6. (a) By finding a bases for the kernels of $A - (-1)I_4$ and $A - 3I_4$, we find that a eigenbasis of $\mathbb{R}^4$ for A is: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Applying the Gram-Schmidt process to these vectors gives the orthonormal basis:

$$\left\{ \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} \\ \frac{-1}{2\sqrt{3}} \\ \frac{3}{2\sqrt{3}} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}.$$

(b) Let S be the matrix whose columns are the vectors in the orthonormal eigenbasis found in (a) (thus S will be an orthogonal matrix), and let D be the matrix whose diagonal entries are the corresponding eigenvalues of A. For example,

$$S = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & 0 & \frac{3}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

(c) S is invertible, so we can write $A = SDS^{-1}$. Thus $A^{100} = (SDS^{-1})^{100} = SD^{100}S^{-1}$.

So,

$$A^{100} = S \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3^{100} \end{bmatrix} S^{-1}$$

$$= \frac{1}{4} \begin{bmatrix} 3+3^{100} & -1+3^{100} & -1+3^{100} & -1+3^{100} \\ -1+3^{100} & -1+3^{100} & -1+3^{100} & -1+3^{100} \\ -1+3^{100} & -1+3^{100} & -1+3^{100} & -1+3^{100} \\ -1+3^{100} & -1+3^{100} & -1+3^{100} & 3+3^{100} \end{bmatrix}.$$

7. (a) The nullclines correspond to the equations $y = 0$ and $y = (x-1)^2 - 1$. The nullcline $y = 0$ is divided into three pieces with, from left to right, directions *up*, *down*, *up*. The nullcline $y = (x-1)^2 - 1$ is also divided into three pieces, with, from left to right, directions *right*, *left*, *right*.

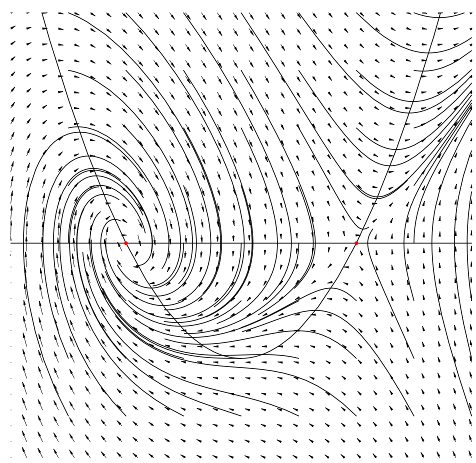
(b) The equilibrium points of this system are at the intesection of the nullclines, and these are the points $(0,0)$ and $(2,0)$. To linearize the system, find the Jacobian matrix:

$$J = \begin{bmatrix} 0 & 1 \\ 2x-2 & -1 \end{bmatrix}$$

$$\rightarrow J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}; J_{(2,0)} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

The eigenvalues of $J_{(0,0)}$ are $\frac{-1 \pm \sqrt{-7}}{2}$. These both have negative real part, so $(0,0)$ is a stable equilibrium. The eigenvalues of $J_{(2,0)}$ are -2 and 1. Since one of these has positive real part, $(2,0)$ is not a stable equilibrium of the system.

(c) For a phase portrait, the directions are as follows: The two nullclines divide the plane into five regions. The direction in the bounded region is *down and left*. For the other four regions, starting with the region below the x-axis and moving clockwise, the directions are *up and left*, *up and right*, *down and right*, *up and right*.



8. (a) $f(x) = \begin{cases} x(x-\pi) & : \quad 0 \le x \le \pi; \\ -x(x+\pi) & : \quad -\pi \le x \le 0. \end{cases}$

$f(x)$ is an odd function, so

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2}}dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0.$$

Now to compute the rest of the coefficients in the Fourier series:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx$$

$$= \frac{2}{\pi}\left[\frac{x(\pi-x)}{n}\cos(nx) + \frac{2x-\pi}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\left[0 + \frac{\pi}{n^2}\sin(\pi n) + \frac{2}{n^3}\cos(\pi n)\right] - \left[0 - \frac{\pi}{n^2}\sin 0 + \frac{2}{n^3}\cos 0\right]\right)$$

$$= \frac{2}{\pi}\left(\frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right)$$

$$= \begin{cases} 0 & : \ n \text{ even;} \\ -\frac{8}{\pi n^3} & : \ n \text{ odd.} \end{cases}$$

Thus the Fourier series for $f(x)$ is:

$$f(x) = \sum_{m=0}^{\infty} -\frac{8}{\pi(2m+1)^3}\sin((2m+1)x).$$

(b) Recall that $\sin((2m+1)\pi/2) = (-1)^m$, so letting $x = \pi/2$ in the solution to (a) gives:

$$f(\pi/2) = \sum_{m=0}^{\infty} -\frac{8}{\pi(2m+1)^3}\sin((2m+1)\pi/2)$$

$$= \sum_{m=0}^{\infty} -\frac{8}{\pi(2m+1)^3}(-1)^m$$

$$\rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = -\frac{\pi f(\pi/2)}{8}.$$

And since we know the value of the function f at $x = \pi/2$, the desired sum is $\pi^3/32$.

(c) As hinted, πx satisfies all but the last condition. And in this condition, we are off by $x^2 - \pi x = x(x-\pi)$. Then as discussed in the Chapter 10 supplement, let

$$R(x,t) = \sum_{m=0}^{\infty} -\frac{8}{\pi(2m+1)^3}\sin((2m+1)x)e^{-(2m+1)^2 t},$$

and notice that $R(x,t)$ satsifies the first three conditions for T as well. Now, if we let $T(x,t) = \pi x + R(x,t)$, we have a function that satisfies all four conditions. So

$$T(x,t) = \pi x + \sum_{m=0}^{\infty} -\frac{8}{\pi(2m+1)^3}\sin((2m+1)x)e^{-(2m+1)^2 t}.$$