

Last Name: _____

First Name: _____

Mathematics 21b

Final Exam
May 22, 2001

Your Section (circle one):

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MWF 10	MWF 11	MWF 11	MWF 12	TuTh 10	TuTh 11:30

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	12	
6	12	
7	14	
8	12	
Total	100	

The exam will last 3 hours.

No calculators are allowed.

Justify your answers carefully (except in Questions 1, 2 and 3).

Write your final answers in the spaces provided.

(1) True or False (no explanation is necessary).

T F : If A and B are 2×2 rotation-dilation matrices then $AB = BA$.

T F : There is a 3×5 matrix A and a 5×3 matrix B with $AB = I_3$ and $BA = 0$ (the zero matrix).

T F : All shears have the same determinant.

T F : If A is a symmetric matrix then the kernel of A is the orthogonal complement of the image of A .

T F : For any matrix A , the product AA^T is diagonalisable.

T F : If A and B are $n \times n$ matrices with A invertible then B and ABA^{-1} have the same eigenvectors.

T F : If all the eigenvalues of A have absolute value less than 1 then $\vec{0}$ is a stable equilibrium for the linear system $\vec{x}(t+1) = A\vec{x}(t)$.

T F : If all the eigenvalues of A have absolute value less than 1 then $\vec{0}$ is a stable equilibrium for the linear system $d\vec{x}(t)/dt = A\vec{x}(t)$.

T F : If $T : C^\infty \rightarrow C^\infty$ is a linear transformation with image C^∞ then T is invertible (in the sense that there is a linear transformation $S : C^\infty \rightarrow C^\infty$ with both ST and TS equal to the identity).

T F : If $T : C^\infty \rightarrow C^\infty$ is a linear transformation with kernel $\{0\}$ then T is invertible (in the sense that there is a linear transformation $S : C^\infty \rightarrow C^\infty$ with both ST and TS equal to the identity).

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- (2) For each of the following matrices and linear transformations circle the value of its determinant. (No explanation is necessary.)

$$-1 \quad \mathbf{0} \quad \mathbf{1} : \left[\begin{array}{ccc} 1 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 3 \end{array} \right]^3.$$

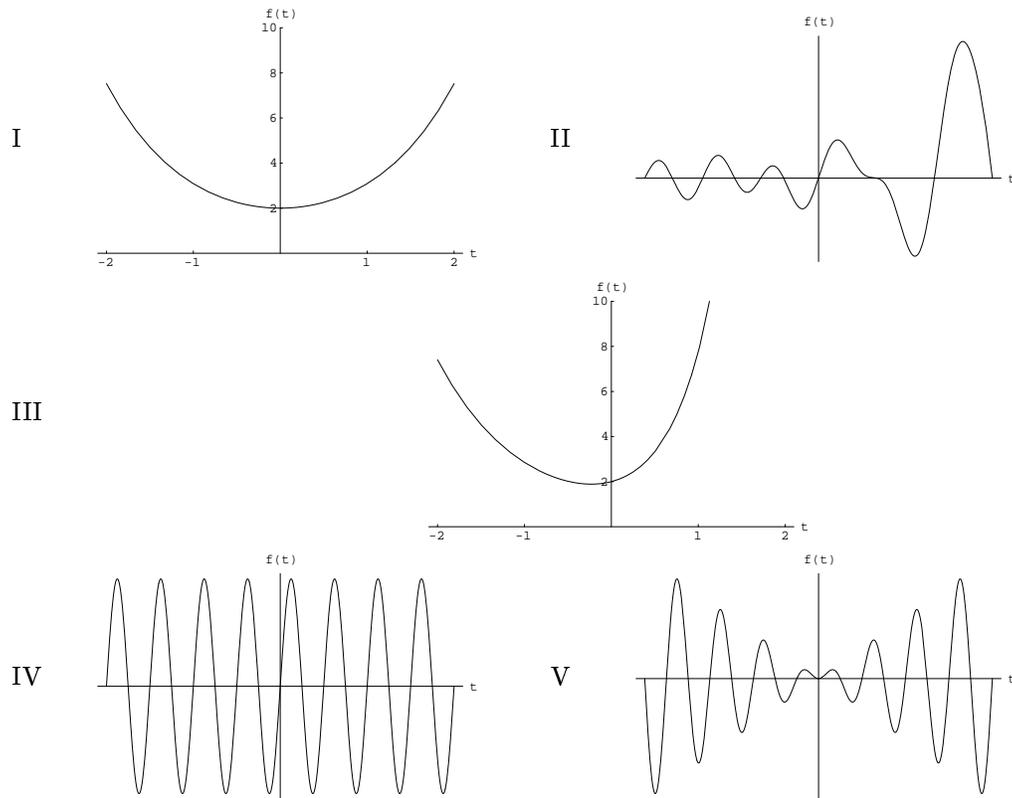
$$-1 \quad \mathbf{0} \quad \mathbf{1} : \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 4 \end{array} \right].$$

$$-1 \quad \mathbf{0} \quad \mathbf{1} : \left[\begin{array}{ccccc} 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$-1 \quad \mathbf{0} \quad \mathbf{1} : \text{Reflection in the plane } x + 2y + 3z = 0 \text{ in } \mathbf{R}^3.$$

$$-1 \quad \mathbf{0} \quad \mathbf{1} : \text{Orthogonal projection to the plane } x + 2y + 3z = 0 \text{ in } \mathbf{R}^3.$$

- (3) Below we have five differential equations and five graphs. Next to each differential equation write the number of the graph that represents a solution to that differential equation. (No explanation is necessary.)



- (a) $df/dt + f = 3e^{2t}$ has a solution represented by graph:
- (b) $d^2f/dt^2 - f = 0$ has a solution represented by graph:
- (c) $d^2f/dt^2 + f = 0$ has a solution represented by graph:
- (d) $16d^2f/dt^2 - 8df/dt + 17f = -16\cos 2t - 47\sin 2t$ has a solution represented by graph:
- (e) $d^2f/dt^2 + f = 2\cos t$ has a solution represented by graph:

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(4) Let $R : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ denote rotation through 120° about the line $x = y = z$ (in a clockwise direction when facing the origin from $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$), let $S : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ denote reflection in the plane $x = y$, and let $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ denote reflection in the plane $y + z = 0$.

(a) Find the matrices for R , S and T (with respect to the standard basis of \mathbf{R}^3).

(b) Find the matrices for RS , ST and RST (with respect to the standard basis of \mathbf{R}^3).

- (c) Give a (precise) geometric description of each of the linear transformations RS , ST and RST .

(5) The matrix

$$B = \begin{bmatrix} 2 & 2 & 2 & 0 & 6 \\ -1 & -1 & -1 & 1 & 0 \\ 2 & 2 & 0 & 2 & 8 \\ -1 & -1 & 0 & 1 & 2 \end{bmatrix}$$

has rank 3. Let E denote the 4×9 augmented matrix $(B|I_4)$.

(a) Find $\text{rref } E$.

(b) Find a basis for the image of B and a basis for the kernel of B .

(c) Find a 5×3 matrix D such that

$$(\text{rref } B)D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d) Find a 3×4 matrix A and a 5×3 matrix C such that $ABC = I_3$.

(6) The matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ has eigenvalues -1 and 3 .

(a) Find an orthonormal eigenbasis of \mathbf{R}^4 for A .

[Space to continue part (a).]

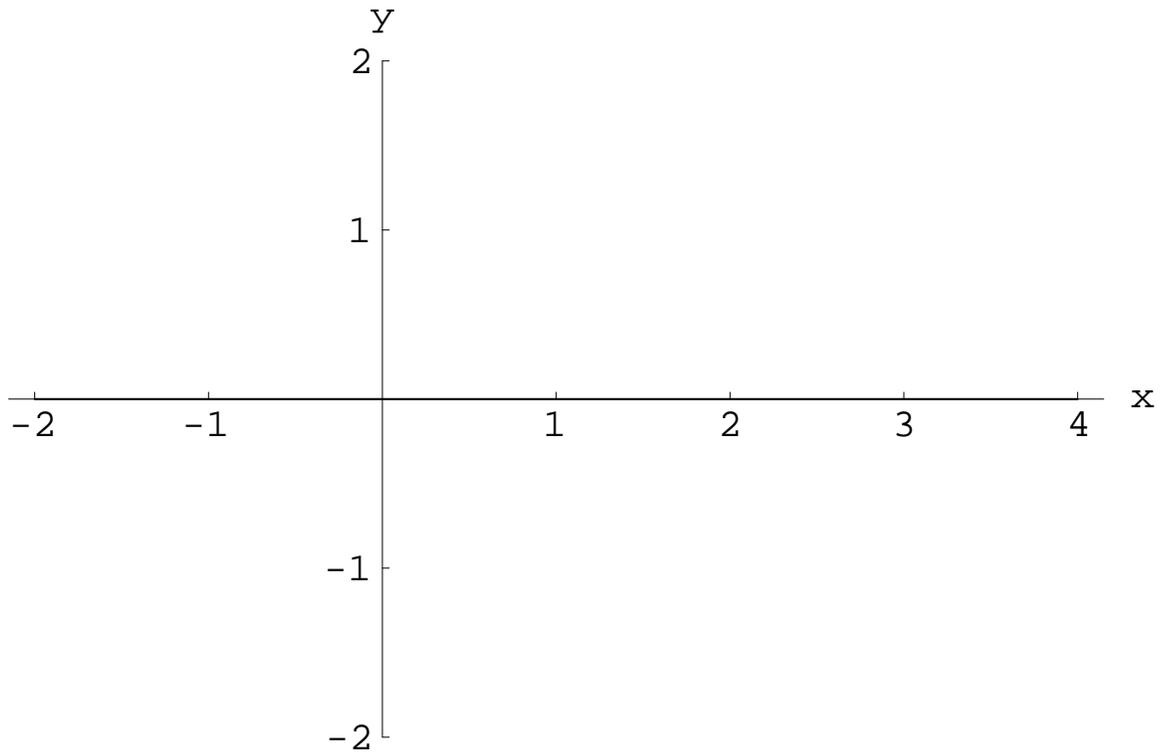
- (b) Find an orthogonal matrix S and a diagonal matrix D such that $AS = SD$.

(c) Calculate A^{100} .

(7) Consider the nonlinear system

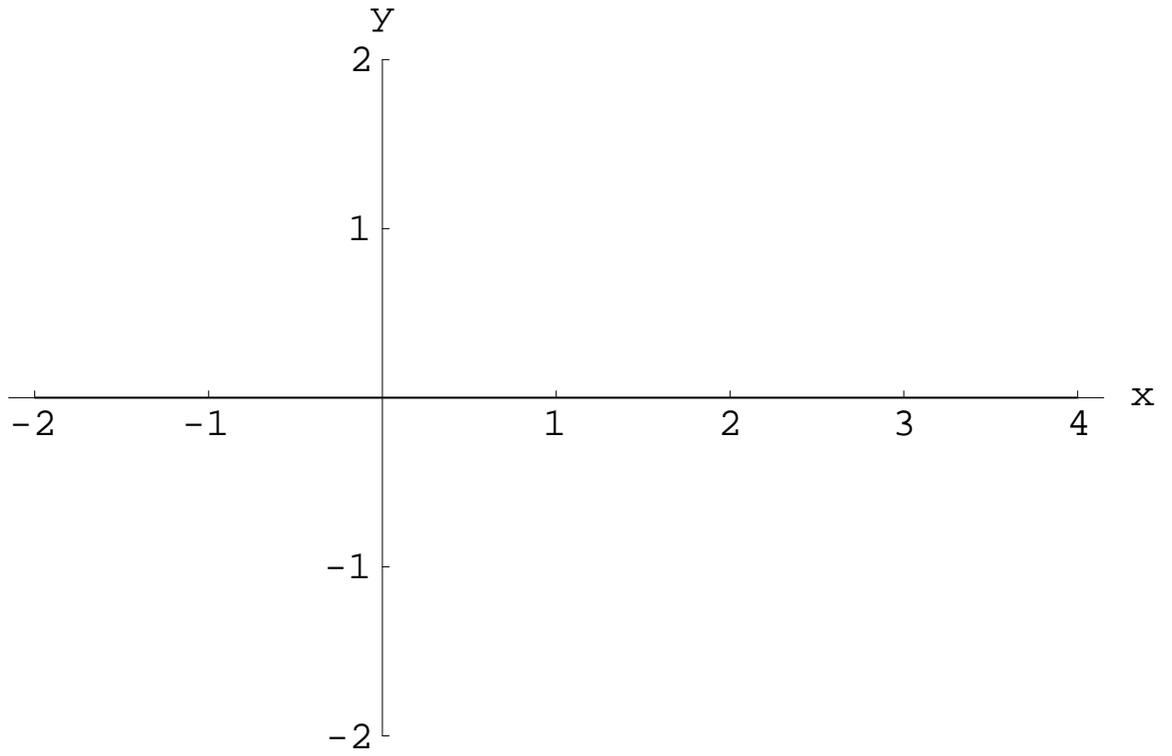
$$\begin{aligned} dx/dt &= y \\ dy/dt &= x^2 - 2x - y. \end{aligned}$$

(a) Sketch the nullclines and a rough direction field for this system.



- (b) What are the equilibrium points? Near each equilibrium point linearise the equations, determine the approximate behaviour and state whether or not that equilibrium point is stable.

(c) Sketch a possible phase portrait.



- (8) (a) Find the Fourier series for the function $f \in C[-\pi, \pi]$ which equals $x(x - \pi)$ in the region $[0, \pi]$ and equals $-x(x + \pi)$ in the region $[-\pi, 0]$.

[HINT: An indefinite integral of $x(x - \pi) \sin nx$ is
 $(x(\pi - x)/n) \cos nx + ((2x - \pi)/n^2) \sin nx + (2/n^3) \cos nx.$]

(b) Compute

$$\sum_{m=0}^{\infty} (-1)^m / (2m+1)^3 = 1 - 1/3^3 + 1/5^3 - 1/7^3 + 1/9^3 - \dots$$

(c) Solve the equation

$$\partial T / \partial t = \partial^2 T / \partial x^2$$

in the region $0 \leq x \leq \pi$, $t \geq 0$ subject to $T(0, t) = 0$, $T(\pi, t) = \pi^2$ and $T(x, 0) = x^2$. [HINT: πx satisfies all but the last of these conditions.]