6. Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

a. Find all real/complex eigenvalues of A with their algebraic multiplicities.

b. Does A have a real/complex eigenbasis? If so, find one.

c. Is A diagonalizable? Why or why not?

d. Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by $T(v) = Av$. Describe T geometrically.

Solution. Notice that A is real symmetric, so it has an eigenbasis of real vectors by the Spectral Theorem. (In fact, the eigenbasis can be taken to be orthogonal.)

a, b. $\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{bmatrix} =$

$$\lambda \det \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} - (-1) \det \begin{vmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{vmatrix} =$$

$$\lambda\left(\lambda(\lambda^2 - 1)\right) + (-1)(\lambda^2 - 1) = (\lambda^2 - 1)(\lambda^2 - 1) =$$

$$(\lambda - 1)(\lambda + 1)(\lambda - 1)(\lambda + 1)$$

$$\boxed{\text{The eigenvalues are } -1 \text{ and } 1\text{, each with algebraic multiplicity } 2.}$$

To find E_{-1}, the eigenspace for $\lambda = -1$, we want to solve

$Av = (-1)v$, or $(-1I - A)v = 0$, so we want $\ker(-1I - A)$.

$$-1I - A = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

so $v \in \ker(-1I - A) \iff \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$, so

$v_1 = -v_4$, $v_2 = -v_3$ and $v = \begin{bmatrix} -v_4 \\ -v_3 \\ v_3 \\ v_4 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are a basis for E_{-1}.

Find E_1: want $\ker(I - A)$

$$I - A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Check that $\ker(I - A)$ has as basis $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

So we have as an eigenbasis $\boxed{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$

with eigenvalues $-1, -1, 1, 1$ respectively.

c. A is diagonalizable because it has an eigenbasis.

d. Note that $A(c_1 e_1 + c_4 e_4) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ 0 \\ 0 \\ c_4 \end{bmatrix} =$

$\begin{bmatrix} c_4 \\ 0 \\ 0 \\ c_1 \end{bmatrix} = c_4 e_1 + c_1 e_4$. So in the $e_1 - e_4$ plane A (or T)

switches the coordinates, so it acts as a reflection about the line $x_1 = x_4$ in the $e_1 - e_4$ plane.

Similarly $A(c_2 e_2 + c_3 e_3) = c_3 e_2 + c_2 e_3$, so it switches coordinates in the $e_2 - e_3$ plane. So it acts as a reflection about the line $x_2 = x_3$ in the $e_2 - e_3$ plane.

7. An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x-y+1) \\ y(y+x-3) \end{bmatrix}.$$

a. Find all equilibrium points of this system in the first quadrant.

b. Sketch the vector field of this system in the first quadrant, indicating the direction of the vector field along the nullclines and inside the regions defined by the nullclines.

c. Are there any stable equilibrium points? Justify your answers.

d. If both species start with positive populations, can either become extinct? Explain.

Solution

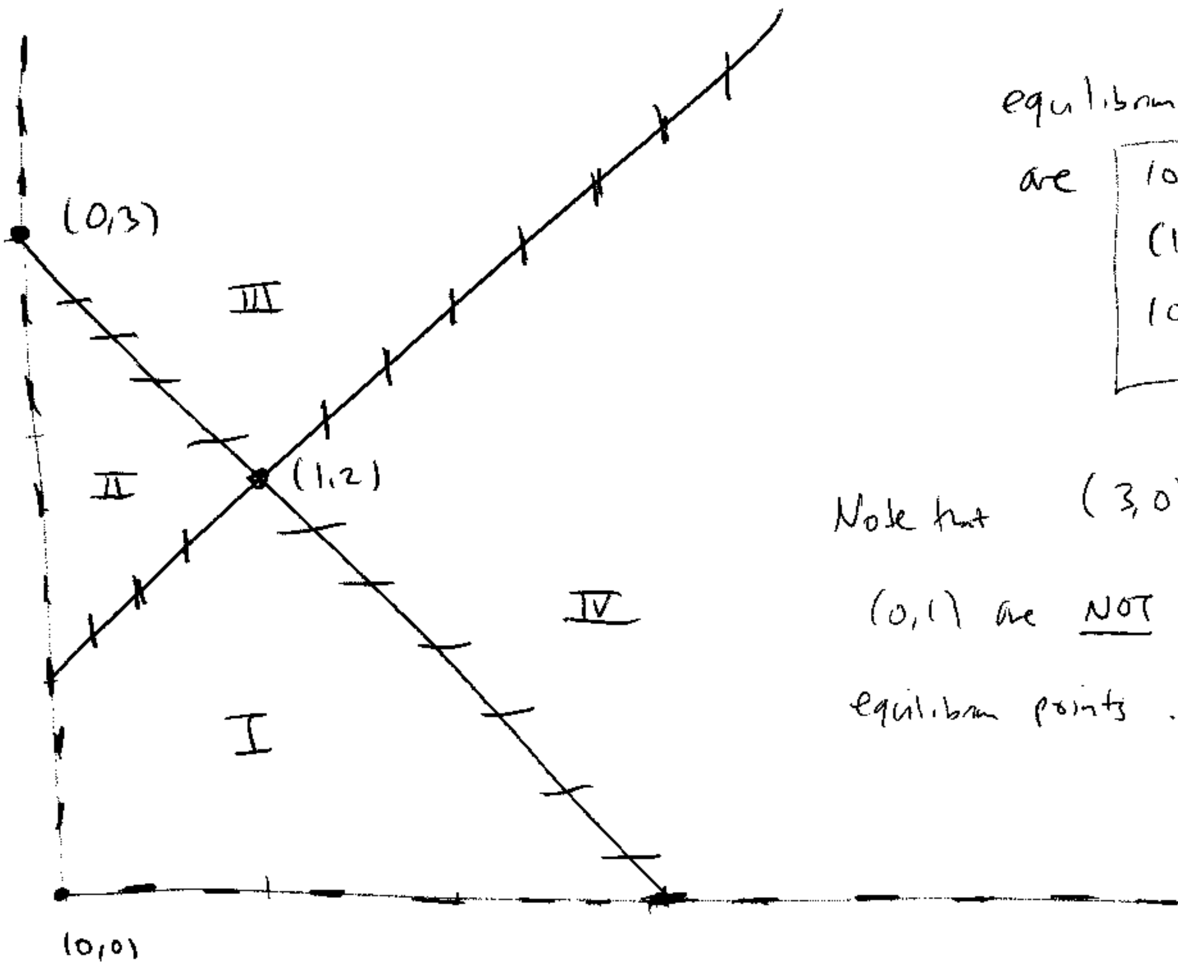
a,b. First we draw the nullclines:

$\frac{dx}{dt} = 0$ (vertical arrows) when $x(x-y+1) = 0$, i.e. when

$(x = 0 \quad \text{OR} \quad x-y+1 = 0)$.

$\frac{dy}{dt} = 0$ (horizontal arrows) when $y(y+x-3) = 0$, i.e. when

$(y = 0 \quad \text{OR} \quad y+x-3 = 0)$

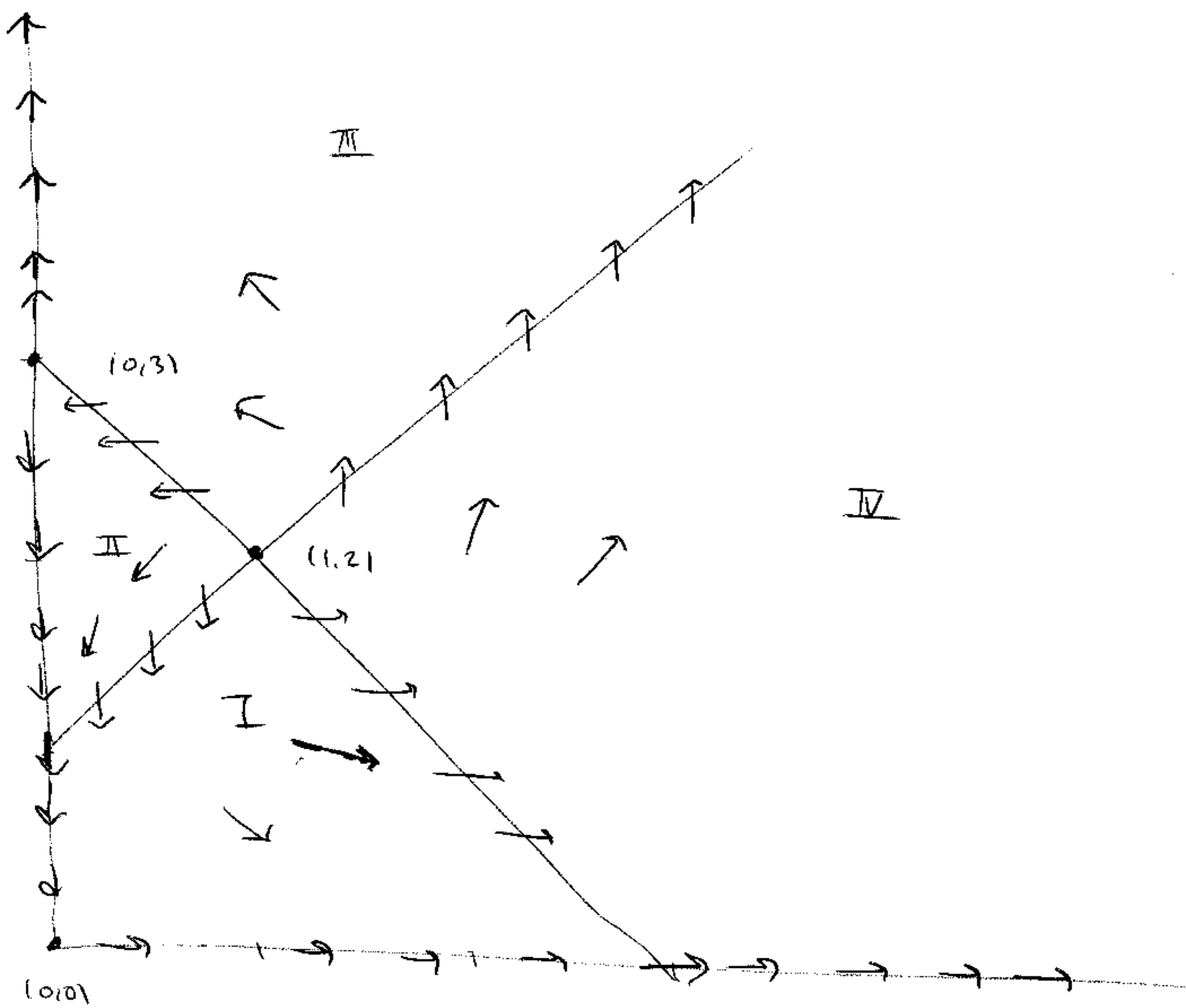


equilibrium points are $\boxed{\begin{matrix}(0,0)\\(1,2)\\(0,3)\end{matrix}}$

Note that (3,0) and (0,1) are <u>NOT</u> equilibrium points.

Now we draw in the arrows:

Region	x	$(x-y+1)$	$\frac{dx}{dt} = x(x-y+1)$	y	$y+x-3$	$\frac{dy}{dt} = y(y+x-3)$
I	+	+	+	+	−	−
II	+	−	−	+	−	−
III	+	−	−	+	+	+
IV	+	+	+	+	+	+



7c. From the picture, it looks like $(0,0)$ and $(0,3)$ are unstable. $(1,2)$ looks like a spiral point, but we don't know if it spirals inward or outward.

To be more precise, we linearize the system near each equilibrium point.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x-y+1) \\ y(y+x-3) \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}.$$

Near an equilibrium point (x_0, y_0), recall that

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \doteq \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}}_{J \,=\, \text{"Jacobian" matrix}} \begin{bmatrix} (x-x_0) \\ (y-y_0) \end{bmatrix}$$

$(0,0)$: $\frac{\partial f}{\partial x} = 2x-y+1 \qquad \frac{\partial f}{\partial y} = -x$

$\frac{\partial g}{\partial x} = y \qquad \frac{\partial g}{\partial y} = 2y+x-3$

so $J = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$. Eigenvalues $1, -3 \Rightarrow \boxed{\text{unstable}}$

$(0,3)$: $J = \begin{bmatrix} -2 & 0 \\ -3 & 3 \end{bmatrix}$ eigenvalues $-2, 3 \Rightarrow$ $\boxed{\text{unstable}}$

$(1,2)$: $J = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ char. poly. is $\lambda^2 - 3\lambda + 4$

eigenvalues λ are $\dfrac{3 \pm \sqrt{-7}}{2}$

which have positive real

part $\Rightarrow$ $\boxed{\text{unstable}}$

7d. Recall that trajectories do not cross or merge at any finite time t. From the picture, we see that if $x=0$ $(y=0)$ at some time t_0, then $x=0$ $(y=0)$ for all times t in the past or future. Hence if both species start with positive populations, neither will go extinct. (Note that it is possible for a trajectory in region $\underline{III}$ to approach the y-axis as $t \to \infty$, so it is possible for population x to approach 0 as $t \to \infty$, but it doesn't actually equal 0 (become extinct) at any time.)