3. Find all solutions to the differential equation

$$f''(t) - 2f'(t) + f(t) = 4e^{3t}.$$

Find the unique solution given the initial conditions

$f(0) = 1$ and $f'(0) = 1$.

Solution. $f = f_h + f_p$ where f_h = solution to the homogeneous equation $f'' - 2f' + f = 0$ and f_p is a particular solution to $f'' - 2f' + f = 4e^{3t}$.

Solutions to $f'' - 2f' + f = 0$: associated polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so solutions are

$$f_h = Ae^t + Bte^t.$$

To find a particular solution f_p to $f'' - 2f' + f = 4e^{3t}$, we guess that $f_p = Ce^{3t}$. Then $f_p'' = 9Ce^{3t}$, $-2f_p' = -6Ce^{3t}$, so we get $4Ce^{3t} = 4e^{3t}$, $C = 1$

so $f_p = e^{3t}$.

The general solution to $f'' - 2f' + f = 4e^{3t}$ is

then $\boxed{f = Ae^{t} + Bte^{t} + e^{3t}, \quad A, B \text{ arbitrary}}$

If $f(0) = 1$, then $1 = A + 1$, so $A = 0$ and

$f(t) = Bte^{t} + e^{3t}$.

If $f'(0) = 1$, we have $f'(t) = Bte^{t} + Be^{t} + 3e^{3t}$,

so $1 = B + 3$, so $B = -2$. Hence

$$\boxed{f(t) = -2te^{t} + e^{3t}}$$

4. Let $V = \{B \in M_n(\mathbb{R}) : B + B^T = 0\}$ be a set of real $n \times n$ matrices. (Recall that such matrices are called <u>anti-symmetric</u>) Show that V is a linear subspace of $M_n(\mathbb{R})$ and find its dimension.

Solution. $0 + 0^T = 0 + 0 = 0$, so 0 is in V.

If B_1 and B_2 are in V, then $B_1 + B_1^T = B_2 + B_2^T = 0$.

Then $(c_1 B_1 + c_2 B_2) + (c_1 B_1 + c_2 B_2)^T =$

$c_1 B_1 + c_2 B_2 + c_1 B_1^T + c_2 B_2^T = c_1(B_1 + B_1^T) + c_2(B_2 + B_2^T)$

$= c_1 \cdot 0 + c_2 \cdot 0 = 0$, so $c_1 B_1 + c_2 B_2$ is in V for any scalars c_1, c_2. Hence V is a linear subspace of $M_n(\mathbb{R})$.

If $B = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}$, then

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & & \vdots \\ v_{1n} & v_{2n} & \cdots & v_{nn} \end{bmatrix} = 0.$$

Hence $2v_{ii} = 0$ for $1 \le i \le n$, so $v_{ii} = 0$.

Also $v_{ij} + v_{ji} = 0$ for $1 \le i,j \le n$, $i \ne j$, so $v_{ji} = -v_{ij}$.

Hence B is of the form $\begin{pmatrix} 0 & v_{12} & v_{13} & \cdots & v_{1n} \\ -v_{12} & 0 & v_{23} & & \\ -v_{13} & -v_{23} & 0 & \cdots & \\ \vdots & & & \ddots & \\ -v_{1n} & & & & 0 \end{pmatrix}$, where

the v_{ij} are arbitrary (for $i<j$). For $i<j$, let e_{ij} be the matrix with a 1 in the ij entry, a (-1) in the ji entry, and zeroes elsewhere. Clearly the e_{ij} are linearly independent.

A typical element B of V is of the form $\displaystyle\sum_{\substack{1 \le i,j \le n \\ i<j}} v_{ij} e_{ij}$,

so the e_{ij} are a basis for V. Now there are

$(n-1) + (n-2) + \cdots + 2 + 1 = \dfrac{(n-1)(n)}{2}$ of these e_{ij},

so $\boxed{\dim V = \dfrac{(n-1)(n)}{2}}$.

5. Let $f \in C[-\pi, \pi]$ be given by $f(x) = \dfrac{e^x - e^{-x}}{2}$.

Determine the Fourier coefficients of f.

Solution. Note that f is odd-symmetric, i.e.

$$f(-x) = -f(x)$$

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2}}\,dx = 0}$$ since $f(x)\frac{1}{\sqrt{2}}$ is odd-symmetric

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0}$$ since $f(x)\cos nx$ is odd-symmetric

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \sin nx - e^{-x}\sin nx\,dx$$

There are two standard ways of computing $\int e^x \sin nx\,dx$.

Method 1: Integration by parts.

$$\int \underset{f'}{e^x}\,\underset{g}{\sin nx}\,dx = (\underset{f}{e^x}\,\underset{g}{\sin nx}) - \int \underset{f}{e^x}\,\underset{g'}{n\cos nx}\,dx$$

$$= (e^x \sin nx) - \left[e^x\, n\cos nx - \int e^x(-n^2)\sin nx\,dx\right]$$

so $\int e^x \sin nx \, dx = e^x \sin nx - n e^x \cos nx$

$- n^2 \int e^x \sin nx \, dx$, so $(n^2+1) \int e^x \sin nx \, dx =$

$e^x \sin nx - n e^x \cos nx$, or $\int e^x \sin nx \, dx =$

$$\frac{e^x \sin nx - n e^x \cos nx}{n^2 + 1}.$$

Method 2. $e^x \sin nx = \operatorname{Im}\left(e^{x+inx}\right) =$

$\operatorname{Im}\left(e^{(1+in)x}\right)$. $\int e^x \sin nx \, dx =$

$\operatorname{Im}\left[\int e^{(1+in)x} dx\right] = \operatorname{Im}\left(\frac{1}{1+in} e^{(1+in)x}\right) =$

$\operatorname{Im}\left[\frac{1-in}{1+n^2}\left(e^x \cos nx + i e^x \sin nx\right)\right] =$

$$\frac{e^x \sin nx - n e^x \cos nx}{1+n^2}$$

Similar methods (or simply a $x \longrightarrow -x$ substitution) show that

$$\int e^{-x} \sin nx \, dx = \frac{-e^{-x} \sin nx - n e^{-x} \cos nx}{1+n^2}.$$

Hence $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x} \sin nx - e^{-x} \sin nx \; dx =$

$$\frac{1}{2\pi} \left[\frac{1}{n^2+1} \left(e^{x} \sin nx - n e^{x} \cos nx + e^{-x} \sin nx + n e^{-x} \cos nx \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n^2+1)} \left(-n e^{\pi} \cos n\pi + n e^{-\pi} \cos n\pi + n e^{-\pi} \cos(-n\pi) - n e^{\pi} \cos(-n\pi) \right)$$

$$= \frac{n}{\pi(n^2+1)} \left(-e^{\pi} \cos n\pi + e^{-\pi} \cos n\pi \right) = \boxed{\frac{(-1)^n n(-e^{\pi} + e^{-\pi})}{\pi(n^2+1)} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx}$$