Solutions to Math 21b Final - Spring 2000

1. Let $A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix}$.

a. Find rref(A), the reduced row-echelon form of A.

b. Find bases for ker(A) and image(A).

c. Find an orthonormal basis for ker(A).

let $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$. Show that $v \in \ker(A)$ and

express v in terms of your orthonormal basis for ker(A)

Solution. a.

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix} \xrightarrow[(III)-9(I)]{(II)+5(I)} \begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & -4 & 8 & -12 \\ 0 & 8 & -16 & 24 \end{bmatrix}$$

$$\xrightarrow{(II)\times -\frac{1}{4}} \begin{bmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 8 & -16 & 24 \end{bmatrix} \xrightarrow[(III)-8(II)]{(I)+2(II)} \boxed{\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

b. $v \in \ker A \Leftrightarrow Av = 0 \Leftrightarrow (\text{rref } A) \cdot v = 0$

so $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$, or $\begin{aligned} x_1 \quad -x_3 + 2x_4 &= 0 \\ x_2 - 2x_3 + 3x_4 &= 0 \end{aligned}$

x_3, x_4 free variables, $x_1 = x_3 - 2x_4$, $x_2 = 2x_3 - 3x_4$.

set $x_3 = s$, $x_4 = t$, then $x_1 = s - 2t$, $x_2 = 2s - 3t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} s - 2t \\ 2s - 3t \\ s \\ t \end{pmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \text{ arbitrary}$$

so $\boxed{\ker A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$. Clearly these vectors are independent and hence a basis for $\ker A$.

Note that rref(A) has leading 1's in the 1st and 2nd columns. Hence image(A) has as a basis the 1st and 2nd columns of A, which are $\boxed{\begin{bmatrix} 1 \\ -5 \\ 9 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 6 \\ -10 \end{bmatrix}}$.

1c. Basis for ker A = $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Use Gram-Schmidt.

Normalize $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$: length is $\sqrt{6}$ so $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}$ is the first vector.

Subtract from $\begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$ the component in the $\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}$ - direction.

$$\begin{pmatrix} -2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix} = \frac{-8}{\sqrt{6}}. \quad \text{so} \quad \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \left(-\frac{8}{\sqrt{6}}\right) \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} \text{ is orthogonal to } \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}.$$

Now normalize: $\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$ has length $\sqrt{\frac{10}{3}}$, so $\sqrt{\frac{3}{10}} \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} =$

$$\begin{bmatrix} \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{30}} \\ \frac{4}{\sqrt{30}} \\ \frac{3}{\sqrt{30}} \end{bmatrix}.$$

$$\boxed{\text{so } \frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ -1 \\ 4 \\ 3 \end{bmatrix} \text{ and } \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \text{ are an orthonorml basis for } Ker(A).}$$

If $v = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$. Check that $\begin{pmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{pmatrix}\begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

so $v \in Ker\, A$. Want c_1, c_2 such that

$$c_1\left(\frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ -1 \\ 4 \\ 3 \end{bmatrix}\right) + c_2\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}.$$

Comparing entries,

get $c_1 \cdot \frac{3}{\sqrt{30}} = 2$ so $c_1 = \frac{2\sqrt{30}}{3}$. Then

$$\frac{2}{3}\begin{bmatrix} -2 \\ -1 \\ 4 \\ 3 \end{bmatrix} + c_2\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \text{ so}$$

$$\left(\frac{2}{3}\right)(-2) + c_2\left(\frac{1}{\sqrt{6}}\right) = 0, \quad -\frac{4}{3} + c_2\left(\frac{1}{\sqrt{6}}\right) = 0, \quad c_2 = \frac{4\sqrt{6}}{3}$$

Hence $\boxed{\begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \frac{2\sqrt{30}}{3}\left(\frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ -1 \\ 4 \\ 3 \end{bmatrix}\right) + \frac{4\sqrt{6}}{3}\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right)}$

2a) We need to show that $A^{-1}J(A^{-1})^T = J$.
We know that $AJA^T = J$, Multiply this equality by A^{-1} from the left hand side and by $(A^{-1})^T$ from the right hand side:

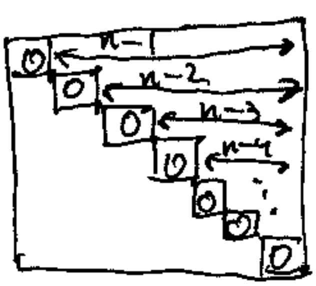
$A^{-1}AJA^T(A^{-1})^T = A^{-1}J(A^{-1})^T$. The expression on the left hand side equals $(A^{-1}A)\,J\cdot(A^{-1}A)^T = J$, so $J = A^{-1}J(A^{-1})^T$, hence A^{-1} is symplectic.

b) $ABJ(AB)^T = ABJB^TA^T = A(BJB^T)A^T = AJA^T = \underline{\underline{J}}$

c) Note that $J^2 = -I$ and $J^T = -J$ so $JJ\cdot J^T = -I\cdot(-J) = J$.

d) Compute $\langle w\cdot v\rangle = w^TJv = (v^TJ^Tw)^T = (-v^TJw)^T = -v^TJw$ (the transposed of a 1×1 matrix v^TJw is itself). So $\langle w,v\rangle = -\langle v,w\rangle$. This is <u>not</u> an inner product as $\langle v w\rangle \neq \langle w v\rangle$.

4) Any antisymmetric matrix has 0's on the diagonal and is competely determined by specifying its entries above the diagonal (because $b_{ij} = -b_{ji}$). Therefore, the dimension of V is equal to the number of such entries which is $(n-1)+(n-2)+\dots+1 = \underline{\underline{\frac{(n-1)n}{2}}}$



6) a) $\det\begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{pmatrix} \overset{(-1)^{1+1}}{=} \lambda\cdot\det\begin{pmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} +$

$+(-1)^{1+4}\cdot(-1)\cdot\det\begin{pmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{pmatrix}$ (Laplace rule).

$= \lambda\cdot\left((-1)^{3+3}\lambda\cdot(\lambda^2-1)\right) + (-1)^{1+3}\cdot(-1)(\lambda^2-1) =$

$= \lambda^2\cdot(\lambda^2-1) - (\lambda^2-1) = (\lambda^2-1)^2 = (\lambda-1)^2(\lambda+1)^2$ so,

$\lambda_1 = \lambda_2 = 1$ (alg # $=2$); $\lambda_3 = \lambda_4 = -1$ (alg # $= -2$).

b) A is a symmetric matrix $\Longrightarrow$ it has a real e/basis. Let us find it. i) $\lambda = 1$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right); \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right);$$

$\begin{cases} x_1 - x_4 = 0 \\ x_2 - x_3 = 0 \end{cases} \Rightarrow$ the basis of the e/space is $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

ii) $\lambda = -1 \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases}$ the basis is

$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}; v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$

<u>Answer</u> v_1, v_2, v_3, v_4 form an e/basis of A

c) Yes, because it has an e/basis

d) Note that $B = \left(\frac{v_1}{\sqrt{2}}, \frac{v_2}{\sqrt{2}}, \frac{v_3}{\sqrt{2}}, \frac{v_4}{\sqrt{2}}\right)$ form an orthonormal basis and $A_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Therefore, A is a reflection with respect to the plane $\underbrace{(w_1 w_2)}_{\text{spanned by}}$.

8) $S(t,x) = t + \frac{x^2}{2}$; $\tilde{T}(t,x) = T(t,x) - S(t,x)$; $\underline{\tilde{T}(t,0) = \tilde{T}(t,\pi) = 0}$

$\underline{\tilde{T}(0,x) = \sin x}$, The solution with such conditions is

$\tilde{T}(t,x) = e^{-t}\sin x$; $T(t,x) = e^{-t}\sin x + t + \frac{x^2}{2}$.

2. Let $J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$.

We say that a 4×4 matrix A is <u>symplectic</u> if

$AJA^T = J$.

a. Show that if A is symplectic, then A^{-1} is symplectic.

b. Show that if A and B are symplectic, then AB is symplectic.

c. Show that J itself is symplectic.

d. Given $v, w \in \mathbb{R}^4$, define $\langle v, w \rangle = v^T J w$. Is this an inner product on $\mathbb{R}^4$? Why or why not?

Solution.

a. Suppose A is symplectic, i.e. $AJA^T = J$. Then

$$
\begin{aligned}
A^{-1}(AJA^T)(A^T)^{-1} &= A^{-1}J(A^T)^{-1} \\
(A^{-1}A)J(A^T(A^T)^{-1}) &= A^{-1}J(A^T)^{-1} \\
J &= A^{-1}J(A^T)^{-1} \\
J &= A^{-1}J(A^{-1})^T
\end{aligned}
$$

so A^{-1} is symplectic.

2b. If A and B are symplectic, then $AJA^T = BJB^T = J$.

Then $(AB)J(AB)^T = ABJB^TA^T = AJA^T = J$, so AB is symplectic.

2c. Direct computation: $JJJ^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = J$ so J is symplectic

2d. Suppose $v = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $\langle v, v \rangle =$

$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = 0.$

But $v \neq \vec{0}$, so $\boxed{\langle\, , \rangle \text{ is not an inner product.}}$