Section 7.6: Stability

8) To determine whether $(0,0)$ is a stable equilibrium of the dynamical system $x(t+1) = Ax$, where $A = \begin{bmatrix} 1 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$, we compute the trace and the determinant and look whether $|\text{tr}(A)| - 1 < \det(A) < 1$. In this case, where $\text{tr}(A) = 1.7, \det(A) = 0.72$, we have stability.

12) The matrix $A = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$ has trace 1.2 and determinant $0.36 + k^2$. The determinant is < 1 for $k < 0.8$, the trace is 1.2, independent of k. Stability requires also $0.2 < 0.36 + k^2$, but that is always satisfied. $\boxed{\text{We have asymptotic stability for } |k| < 0.8.}$ We can also see this from the fact that the matrix is a rotation-dilation matrix which corresponds to a complex multiplication with $0.6 + ik$ which is a complex number $|\lambda| < 1$ for $|k| < 0.8$.

22) To find a closed form solution of $x(t+1) = Ax$ with $A = \begin{bmatrix} 7 & -15 \\ 6 & 11 \end{bmatrix}$, we find the eigenvalues and eigenvectors, express $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a sum of eigenvectors $\vec{v} = a\vec{v}_1 + b\vec{v}_2$ and have the solution $\vec{v}(t) = a\vec{v}_1 e^{\lambda_1(t)} + b\vec{v}_2 e^{\lambda_2(t)}$. In our case, $\lambda_1 = -2+3i, \lambda_2 = -2-3i$, $\vec{v}_1 = \begin{bmatrix} (3+i)/2 \\ 1 \end{bmatrix}$. $\vec{v}_2 = \begin{bmatrix} (3-i)/2 \\ 1 \end{bmatrix}$ and $\vec{v} = a\vec{v}_1 + b\vec{v}_2$, with $a = (1+3i)/2, b = (1-3i)/2$.

28) If $x(t+1) = Ax$ has a stable origin, then the eigenvalues of A satisfy $|\lambda| < 1$. The matrix A has the eigenvalues $\lambda + 2$. They can no more satisfy $|\lambda| < 1$ and the origin is unstable.

40) a) $A^T A = (p^2 + q^2 + r^2 + s^2) I_4$.
b) Because the kernel of A is equal to the kernel of $A^T A$, the matrix A is invertible if and only if $p^2 + q^2 + r^2 + s^2 \neq 0$. c) $\det(A^T A) = \det(A)^2 = (p^2 + q^2 + r^2 + s^2)^4$ so that $\det(A) = (p^2 + q^2 + r^2 + s^2)^2$.
d) $\det(A - \lambda) = ((p-\lambda)^2 + q^2 + r^2 + s^2)^2 = 0$. So, $\lambda = p + i\sqrt{q^2 + r^2 + s^2}$.
e) $|Ax| = |\lambda||x|(p^2 + q^2 + r^2 + s^2)$.
f) Take a first quaternion matrix A defined by $(p_1, q_1, r_1, s_1) = (3, 3, 4, 5)$ then a second quaternion matrix B defined by $(p_2, q_2, r_2, s_2) = (1, 2, 4, 4)$. The product AB is again a quaternion matrix defined by (p, q, r, s). The corresponding vector is $39, 13, 18, 13$. Indeed $39^2 + 13^2 + 13^2 + 18^2 = 2183$.
g) If $n = p_1 p_2 \cdots p_n$ is the primefactorization, and we know the sum of the squares for each prime, then we can form the corresponding quaternion matrices A_i and form the matrix $A = A_1 ... A_n$ which is defined by some (p, q, r, s).

38) a) $Aw + b = w$ means or $(A - 1)w = -b$ or $w = (A-1)^{-1} b$.
The affine transformation can be modeled by a linear transformation $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. The eigenvalues of B are the eigenvalues of A and 1. If all eigenvalues of A are smaller then 1, then $T^n \vec{v}$ converges to $\vec{w}$.

42) a) We can write the system as $T(x, y) = (x - ky, y + k(x - ky)) = (x - ky, kx + (1-k)y)$.

The corresponding matrix is $A = \begin{bmatrix} 1 & -k \\ k & (1-k^2) \end{bmatrix}$.
b) The matrix A has all eigenvalues of length 1: $\det(A) = 1$, $\mathrm{tr}(A) = 2 - k^2 < 2$.

Section 8.1: Symmetric matrices
2) $[1,1], [1,-1]$ are eigenvectors to the eigenvalues $2, -2$. Normalize them to get an orthonormal basis.

10) The eigenvalues are 0 and 9. An eigenvector to 9 is $[1,-2,2]$, eigenvectors to 0 are $[1,-2,2]$ and $[-2,0,1]$. Normalize them to get an orthonormal eigenbasis.

12) a) Take $v = [1,0,2]/\sqrt{5}$, $u = [2,0,-1]/\sqrt{5}$ and $w = [0,1,0]$.
b) The matrix is $B = Diag(-1,1,1)$.
c) Form the matrix S which has the vectors v, u, w as columns and form $A = SBS^{-1}$:

$$S = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix}, A = SBS^{-1} = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{bmatrix}$$

16) a) $[1,1,1,1,1]$ is the eigenvector to the eigenvalue 5. The four eigenvectors to the eigenvalue 0 are $[1,-1,0,0,0], [1,0,-1,0,0], [1,0,0,-1,0], [1,0,0,0,-1]$.
b) Note that $B = A + 2$. It has the eigenvalues $7, 2$ with the same eigenvectors.
c) The determinant of B is $7 * 2 * 2 * 2 * 2 = 112$.

36) Yes, A has the eigenvalues 0 and 1. When diagonalized, it is a projection.

24) The characteristic polynomial is $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2$. We can find eigenvectors $[1,0,0,1], [0,1,1,0]$ to the eigenvalues 1 and $[-1,0,0,1], [0,-1,1,0]$ to the eigenvalues -1.

26) The matrix interchanges e_n with e_1, e_{n-1} with e_2, etc. In the case, when $n = 2m$ is even, this can be seen as the product of m reflections in planes. There are then m eigenvalues 1 and m eigenvalues -1. In the case, when $n = 2m+1$ is odd, then there are $m+1$ eigenvalues 1 and m eigenvalues -1.