Section 7.3 : Finding the eigenvalues of a matrix

14) The characteristic polynomial is $x^3 - 2x^2 - x = x(x-1)^2$. The matrix has a complete eigensystem with eigenvectors $[-2,0,5],[-1,5,0]$ to the eigenvalue 1 and $[0,1,0]$ to the eigenvalue 0.

20) The eigenvalues 1 has algebraic multiplicity 2. The eigenvalue 2 has algebraic multiplicity 1. The geometric multiplicity of the eigenvalue 2 does not depend on a,b,c. Look at $\ker(A-1) = \ker(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 2 \end{bmatrix}$. You see that this is two dimensional if $a=0$ and one dimensional if $a \neq 0$. Therefore, for $a=0$, the geometric multiplicity of 1 is 2 and we have an eigenbasis. For $a \neq 0$, the geometric multiplicity of 1 is 1 and we have no eigenbasis.

34) a) $B\vec{x} = \vec{0}$ implies $S^{-1}AS\vec{x} = \vec{0}$. This implies $AS\vec{x} = \vec{0}$ because S^{-1} has only the trivial kernel.
b) The map has an inverse. S^{-1} maps the kernel of B to the kernel of A.
c) The dimension of the kernel is left the same as well as the dimension of the image by the dimension formula.

42) From $C(t+1) = 0.8C(t) + 10$, we get

$$A = \begin{bmatrix} 0.8 & 10 \\ 0 & 1 \end{bmatrix} .$$

This matrix has an eigenvector $\vec{v}_1 = [50,1]$ to the eigenvalue 1 and an eigenvector $\vec{v}_2 = [1,0]$ to the eigenvalue 0.8. The vector $\vec{v} = [0,1]$ can be written as $\vec{v} = \vec{v}_1 - 50\vec{v}_2$ and so $A^n\vec{v} = A^n\vec{v}_1 - 50A^n\vec{v}_2 = \vec{v}_1 - 50(0.8)^n = [50(1-0.8^n),1]$. Therefore $C(t) = 50(1-4^t/5^t)$. The function is 0 at $t=0$, increases monotonically for increasing t and reaches asymptotically, the value 50.

48) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue $\lambda_1 = (1+\sqrt{5})/2$ with eigenvector $\vec{v}_1 = [\lambda_1, 1]$ and the eigenvalue $\lambda_2 = (1-\sqrt{5}/2$ with eigenvector $\vec{v}_2 = \lambda_2, 1]$. Because $[1,0] = (\vec{v}_1 - \vec{v}_2)/\sqrt{5}$, we know $A^n\vec{v} = (A^n\vec{v}_1 - A^n\vec{v}_2)/\sqrt{5} = (\lambda_1^n\vec{v}_1 - \lambda_2^n\vec{v}_2)/\sqrt{5}$. So $j(n) = (\lambda_1^n - \lambda_2^n)/\sqrt{5}, a(n) = j(n+1)$. In the limit $n \to \infty$, the term λ_2^n vanishes and we have $j(n+1)/j(n) \to \lambda_1$, the golden ratio.

38) The characteristic polynomial has degree 3 and has a real root. Because rotations preserve length, each eigenvalue must have absolute value 1. Therefore, $\lambda = 1$ or $\lambda = -1$. Note that this does not exclude $\lambda = -1$ and indeed, $\lambda = -1$ can also occur for a rotation matrix. If all eigenvalues are real, then they have all to be 1 or two of them have to be -1 because the product of the eigenvalues is the determinant which is 1. If two of the eigenvalues are complex, then their product is the square of the length of these numbers which is 1. Again, because the determinant is 1, the third real eigenvalue is 1.

Section 7.4 : Diagonalization

14) The matrix has three different eigenvalues and is therefore diagonalizable. The eigenvectors are $[1,0,0],[-1,1,0],[0,-1,1]$. The matrix S, diagonalizing A is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

18) The matrix has the eigenvalues $0,1,2$, which are different, so A is diagonalizable. The eigenvectors are $[1,0,1],[0,1,0],[-1,0,1]$. The matrix S diagonalizing A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

22) For $b=1$, the matrix is diagonalizable if and only if $a \neq 0$. For $b \neq 1$ the matrix is always diagonalizable.

36) The first matrix has the eigenvalues $2,3$ and both have the same trace and determinant so that both matrices have the same eigenvalues. Because both matrices are diagonalizable, they are similar.

54) If they were, also A^2 and B^2 were similar but $A^2 = 0$ and $B^2 \neq 0$. The matrices are not similar.

58) This is the Cayley-Hamilton theorem. Since it is true for the diagonalized matrix B, we apply S to the left and S^{-1} to the right of the identity $f_A(B) = 0$ to get also $f_A(A) = 0$.

56) The characteristic polynomials of both sides in the hint are the same. Because of the product property of the characteristic polynomial, the characteristic polynomial of the left hand side is the one of AB, while the characteristic polynomial of the right hand side is the one of BA.

Section 7.5 : Complex Eigenvalues

6) If $z = re^{i\phi}$, then $1/z = r^{-1}e^{-i\phi}$.

12) The characteristic polynomial of the complex conjugate $\overline{A}$ is $\overline{f}_A(\lambda)$, the polynomial, where all coefficients are conjugated. But this is the same as the characteristic polynomial of A because A is real. Since $f_A(\lambda) = \prod_i(\lambda - \lambda_i) = \overline{f}_A(\lambda) = \prod_i(\lambda - \overline{\lambda_i})$, the fundamental theorem of algebra assures that the set $\{\overline{\lambda}_i\}$ is the same as the set $\{\overline{\lambda}_i\}$.

24) The characteristic polynomial is $\lambda^3 - 3\lambda^2 + 7\lambda - 5$ which has three roots $1, 1+2i, 1-2i$. There is an eigenvector $[1, 1, 1]$ to the eigenvalue 1. The eigenvector to $1 + 2i$ is $[-3 + 4i, 5 + 10i]$, the eigenvector to $1 - 2i$ is $[-3 - 4i, 5 - 10i]$.

30) a) Write out the matrix product: $A\vec{x} = \vec{y}$.
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1$,
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2$,
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3$.
Summing up these equations gives $x_1 + x_2 + x_3 = 1$. This computation actually shows that if one multiplies two Markov matrices A, B, then also the product is a Markov matrix (use the same argument where $\vec{x}$ is a column vector of B).

b) Since all eigenmodes to eigenvalues with absolute value < 1 will die out, only the eigenvector to the eigenvalue 1 will survive. If the initial vector $\vec{v}$ is of the form $\vec{v} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$, where $\vec{v}_1$ is the eigenvector to 1, then $A^n\vec{v} \to a_1\vec{v}_1$.

36) a) Look at the column vectors. The first column gives the age distribution in 15 year, if we would start with a population consisting of 0-15 year only. 82 percent of this population will get into the 15-30 year group, the group itself will have 110 percent. The group of 15-30 year old will produce 0.6 time its size in 0-15 year olds, 19 percent will die. etc.

b) The maximal eigenvalue is 1.9 with eigenvector
$\vec{v} = [0.900674, 0.387094, 0.180568, 0.0766587, 0.0212947, 0.00323672]$. The eigenvector describes the stable population structure. If we rescale $\vec{v}$ so that the sum of its entries is 1, then we get the percentage distribution of the different age groups:
$[0.573851, 0.246631, 0.115046, 0.0488419, 0.0135676, 0.00206223]$.
This means for example that 57 percent of the stable population has age 0-15.