Section 6.3 Geometric Interpretation Cramers rule

4. The area of the triangle is one half of the area of the parallelogram which is

$$A = \det(\begin{bmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{bmatrix})/2 \;.$$

This is the same as $\det(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix})/2$ which is the volume of a prism over the triangle, half of a parallelepiped.

12. With the given entries, we can build vectors of length 4 maximally. The four dimensional volume of the parallelepiped spanned by 4 vectors is maximal if all four vectors are orthogonal. In four dimensions, we can find four orthogonal vectors of length 2 with entries ± 1 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

14. The volume is $\sqrt{\det(A^T A)}$ with $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, which is $\sqrt{6}$.

22. $x = \det(\begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix})/\det(\begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}) = -10/5 = -2$.
$y = \det(\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix})/\det(\begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix}) = 5/5 = 1$.

26) Yes, the explicit formula for A^{-1} involves determinants of submatrices of A which are integers. The only factor in the denominator is the determinant of A which is an integer by assumption.

18)
a) The circle is stretched in the x and y directions. The area changes by the determinant of A which is pq. So, the area is πpq. We know therefore the area of the region inside the ellipse.

b) Because the area of the ellipse is πab and this is $\det(A)\pi$ we know $ab = \det(A)$.

31) Let $r = 1/(2\sqrt{3})$ be the radius of the circle inscribed an equilateral triangle of length 1. The area of the equilateral triangle is $3r/2$, the area of the circle $r^2\pi$. The relation is $C = r^2 2\pi/(3r) = r\pi 2/3 = \pi/(3\sqrt{3}) \sim 0.6$. Because the new triangle has area 6, the inscribed ellipse has aerea $6C = 2\pi/\sqrt{3}$.

Section 7.1 Eigenvalues and Eigenvectors

2) $A\vec{v} = \lambda\vec{v}$ implies $\vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v}$ so that $\lambda^{-1}\vec{v} = A^{-1}\vec{v}$ and λ^{-1} is an eigenvalue of A^{-1}.

10) Write down what it means and get a system of equations $a + 2b = 5, c + 2d = 10$. We can chose for example a, c freely and get b, d from these equations.

20) For any rotation in space the axes of rotation contains an eigenvector to the eigenvalue 1.

38ab) From the description we know the images of the basis vectors and so the columns of A: $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, a matrix, which indeed has the given eigenvectors $\vec{v}_1$ with eigenvalue 2 and $\vec{v}_2$ with eigenvalue -1.

c) Because $\vec{e}_1 = (\vec{v}_1 + \vec{v}_2)/3$, we have $A^n\vec{e}_1 = (A^n\vec{v}_1)/2 + (A^n\vec{v}_2)/3 = 2^n\vec{v}_1/3 + (-1)^n\vec{v}_2/3 = \begin{bmatrix} 2^n/3 + 2(-1)^n/3 \\ 2^n/3 - 1(-1)^n/3 \end{bmatrix}$.

40) The matrix has the eigenvalue $5^2 + 8 + 3 = 27$ with the same eigenvector.

Section 7.2 Finding the eigenvalue

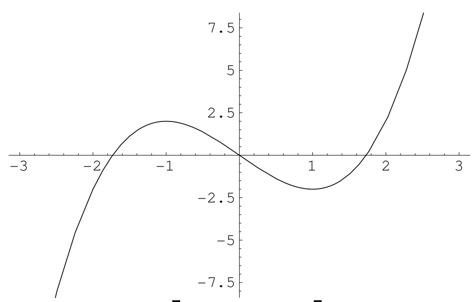
8) The characteristic polyonomial is $x^3 - 3x^2 = x^2(x-3)$ which shows that there are two eigenvectors to the eigenvalue 0 and one eigenvector to the eigenvalue 3. The eigenvector to 3 is $[1,1,1]^T$, the eigenvectors to 0 are spanned by $[1,-,1,0],[0,1,-1]$. Geometrically, A is a projection onto the plane $x+y+z=0$ followed by an dilation by a factor 3.

10) The matrix has the eigenvalues -1 with algebraic multiplicity 2 and geometric multiplicity 1: the eigenvector is $[2,0,1]^T$. The eigenvalue 1 has the eigenvector $[1,0,1]^T$.

16) The eigenvalues are $T/2 + \sqrt{(T/2)^2 - D}$ where T is the trace and D is the determinant. There is one eigenvalue if $T^2 = 4D$, two real eigenvalues if $T^2 > 4D$ and no real eigenvalue if $T^2 < 4D$.

22) The characteristic polynomials are the same because $\det(A) - \det(A^T)$ and so $\det(A-\lambda) = \det(A^T - \lambda)$.

32) The characteristic polynomial is $\lambda^3 - 3\lambda + k$. We have to find the roots $\lambda^3 - 3\lambda = -k$. This polynomial has a local maximum 2 at -1 and a local minimum -2 at 1. For $k \in (-2,2)$, there are three solutions, for $k \in \{-2,2\}$ there are two roots (one with algebraic multiplicity 2) and for $|k| > 2$, there is only one root (the other roots are complex).



28) a) $A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ which has an eigenvalue 1 with eigenvector $[1,2]^T$. There is an other eigenvalue 0.7 with eigenvector $[-1,1]^T$.

b) We have $[1200,0]^T = 400[1,2]^T + 800[1,-1]^T$ so that $[w(n), n(t)]^T = 400[1,2]^T + 8000.7^n[1,-1]^T$.

c) As $n \to \infty$, the situation stabelizes at $[400, 800]$, so that also the family store survives the supermaket assault.

38) a) Because $x^3 + 6x$ has a positive derivative, the function is monotone and invertible.
b) Cardano claims that if $v - u = x, uv = 2v^3 - u^3 = 20$, then x solves $x^3 + 6x = 20$. Indeed, $x^3 = (v-u)^3 = v^3 - u^3 - 3v^2u + 3vu^2 = 20 - 3vu(u-v) = 20 - 6x$.
c) From the second equation get $u = 2/v$, then get $v^3 - 8/v^3 = 20$ which has the real solutions $v = 1 - \sqrt{3}$ and $v = 1 + \sqrt{3}$ and $u = (-1+\sqrt{3})$ or $(-1-\sqrt{3})$. Therefore, $x = v - u = 2$
d) Same computation but with constants. If p is negative, then step a) goes wrong.
e) Just plugin in $x = t - a/3$ into $x^3 + ax^2 + bx + c$ to get $t^3 + (b - a^2/3)t + (2a^3/17 - ab/3 + c)$ which is the Cardano form.