Section 5.3 Projection

2) We use that $A^T A = I_n$ and $\vec{v} \cdot A\vec{w} = A^T\vec{v} \cdot \vec{w}$ so conclude

$$A\vec{v} \cdot A\vec{w} = A^T A\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{w} \; .$$

This is an important property of orthogonal transformations. It implies that both and length are preserved if $A^T A = I_n$.

6) Yes, we have to show that if $A^T A = I_n$, then also $AA^T = B^T B = I_n$, where $B = A^T$.

From $A^T A = I_n$, we get $A^T = A^{-1}$ which can be read as $B = (B^T)^{-1}$. Taking inverses shows $B^{-1} = B^T$. By multiplying both sides from the right with B, we end up with $B^T B = I_n$.

(It is useful to remember from this that also the rows of an orthogonal matrix form an orthormal basis.)

8) a) No, take just the example on the same page, where $A^T A$ is I_2 while AA^T is a projection matrix.

b) Yes, it is the case because we have seen in problem 6) that $B = A^T$ is also $AA^T = I_n$.

16) Yes: $A = A^T$ implies $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$.
Note that if A, B are different symmetric matrices, then AB is not necessary symmetric any more.

20) The two vectors are not yet orthonormal, but a Gram-Schmidt orthonormalisation gives to normal vectors which can be used to define $A = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix}$ and get the projection

$$P = AA^T = \begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix} /50.$$

18) a) $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
b) Actually, A^2 is symmetric, because $(A^2)^T = (AA)^T = (-A^T)(-A^T) = (A^T)^2$.

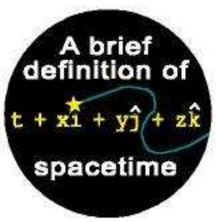
44) We can write any quaternion matrix as $M = p1 + qi + sj + rk$, where

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, j = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

From this is is obvious that M is a linear space of dimension 4. Because $i.j == -i.j = k, j.k = -k.j = i, k.i = -i.k = j, i.i = j.j = k.k = -1$, the product of quaterion matrices is a quaternion matrix. We compute $M^T M = (p^2 + q^2 + r^2 + s^2)I_4$ and see from this that M is invertible if $(p^2 + q^2 + r^2 + s^2)$ is not zero. In the other case, we have the zero matrix. The inverse is in

M too because $M^{-1} = M^T/(p^2+q^2+r^2+s^2)$ and $M^T = -M + pI_4$ is a quaternion matrix too. Quaternions do not commute because $ij = -ji$.

ROTATION WITH QUATERNIONS. One can calculate with quaternions directly without the matrix representation. One often writes $q = (s, \vec{p}) = s + ip_1 + jp_2 + kp_3$. Quaternions are useful to compute rotations in space. If one wants to rotate a vector $\vec{v}$ in space by an angle ϕ around an axis which contains the unit vector $\vec{u}$, one can form the quaternions $q = (\phi, \vec{u})$ and $p = (0, \vec{v})$ and form the new quaternion $p' = qpq^{-1}$. It has the form $p' = (0, \vec{v}')$, where $\vec{v}'$ is the rotated vector. This algebraic manipulation is useful in physics or computer graphics.



DISCOVERY. Quaternions were discovered by William Rowan Hamilton while walking along the Royal Canal. He was so excited about his "invention" that he wrote the properties $i^2 = j^2 = k^2 = ijk = -1$ of the quaternions into the stone of the Brougham bridge. Quaternions are also called **hypercomplex numbers**. They are not only used in computer graphics, also physisists find them handy.

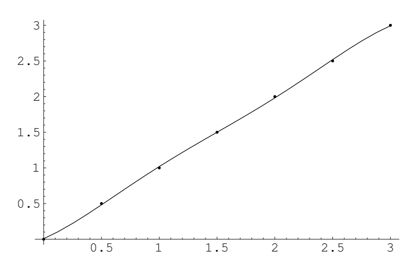
Section 5.4 Orthogonality and least squares

2) The kernel of A^T is equal to the orthogonal complement of the image of A. Since the later is a plane, the kernel of A^T is a line. To find the kernel of $B = A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, we do row reducution and get $\text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ from which we see that $[1, -2, 1]^T$ is a basis for the kernel.

10) a) Consistent means that we have a solution $A\vec{v} = \vec{b}$. The hint leads the path: split up this solution $\vec{v} = \vec{v}_h + \vec{v}_0$, where $\vec{v}_h$ is in the kernel of A and $\vec{v}_0$ in the complement. Now, $A\vec{v}_h = 0$ implies $A\vec{v}_0 = \vec{b}$ also and we found the solution.
b) If there were two solutions x_0, y_0 in the orthogonal complement of the kernel, then the element $\vec{v} = \vec{x}_0 - \vec{y}_0$ in the orthogonal complement of the kernel satisfies $A\vec{v} = \vec{0}$. But because $\vec{v}$ is both in the kernel and the orthogonal complement, it is $\vec{0}$ and $\vec{x}_0 = \vec{y}_0$.
c) This follows from Pythagoras and the relation $\vec{x}_1 = \vec{x}_0 + \vec{x}_h$.

22) Use the routine formula $(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

34) $f(t) = 1.5 - 1.53669\cos(t) + 0.0431477\cos(2t) + 0.108974\sin(t) + 0.302692\sin(2t)$ It was of course ok to use technology here.



40) We fit the data poits $(x_i, \log(y_i))$ with respect to the functions $1, t$ and get the best linear fit $f(s) = 0.171666 + 0.155262s$. This means that $g(t) = 1.18728e^{0.155252t} = 1.187281.167976^t$ is the best fit.

16) Assume A is a $m \times n$ matrix so that A^T is a $n \times m$ matrix Apply dimensions to $\text{im}(A)^\perp = \ker(A^T)$ gives $m-\text{rank}(A) = m-\text{rank}(A^T)$, where we have used that $\text{dimker}(A^T)+\text{dimran}(A^T) = m$.

18) The rank of A^TA is smaller or equal then the rank of A. Because the kernel of A^T is perpendicular to the image of A, the rank of A^TA is indeed equal to the rank of A. and because the ranks of A and A^T agree also the rank of A^TA is equal to the rank of AA^T.

Section 6.1 Determinants

14) This is a permutation matrix. There are 6 inversions so that the determinant is 1.

4) Do Laplace expansion with respect to the first row, then with respect to the new 3'd row. The determinant is $39\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 7 & 4 & 1 \end{bmatrix} = -324$.

6) After row reduction, end up with a matrix M_{n-1}. Because $M_1 = 1$, we have $M_n = 1$ for all n.

12) Use the linearity to see that we have the same determinant as A which is 8.

16) We have $t^2(b-a)$. Because for $a = t$ and $b = t$, the matrix is not invertible, we see that $f(t) = c(t-a)(t-b)$ and from a) we get $f(t) = (a-b)(t-a)(t-b)$. The matrix is invertible if $a \neq b$ and t is different from both a and b.

40) The function $f(t)$ is a polynomial of degree 4. Now $f(0) = (\pm 1)^2 - 1 = 0$ and similarily $f(1) = f(2) = f(3) = f(4) = 0$ and consequently by the fundamental theorem of algebra, $f(t) = 0$ for all t. Therefore, $\det(A) = \pm 1$ which implies that the inverse of A has integer entries.