

Section 3.3 Dimension.

22) $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are three Pivot columns, so that columns 1, 3, 5 of

the original are a basis of the image of A . The kernel is 2-dimensional. We can introduce free

variables s, t, u for columns 2, 4. and get the kernel spanned by $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

24) Form

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where A contains the given vectors as columns. The first and third columns are pivot columns.

Therefore, the first and third vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{bmatrix}$ span the subspace.

32) We look for the kernel of the matrix $A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ which is already in row reduced form. The first two columns are pivot columns. Attach free variables to the last two columns so that $w = t, z = s, x = s - t, y = -2s - 3t$, so that a general element in the kernel is

$$\vec{x} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

38) a) Because the image has dimensions 0, 1, 2 or 3, the kernel has by the dimension formula the dimensions 5, 4, 3 or 2.

b) The image of T has maximal 4 dimensions (look at the matrix in row reduced echelon form, there can be maximal 4 leading 1 and therefore maximally 4 pivot columns). The possible values of the rank of T are 0, 1, 2, 3, 4.

52) Transpose the matrix and row reduce. The first and third column of the transposed matrix A^T are pivot columns so that the first and third row of A form a basis of the row space.

36*) No, this is not possible by the dimension formula. The dimensions of the image and kernel have to add up to 3.

56*) The hint gives the solution away already. Assume these vectors are linearly dependent, then one could have

$$c_0 \vec{v} + c_1 A \vec{v} + \dots + c_{m-1} A^{m-1} \vec{v} = \vec{0}.$$

Multiplying both sides with A^{m-1} using $A^m = 0$ shows that $c_0 = 0$. We are left with

$$c_1 A \vec{v} + \dots + c_{m-1} A^{m-1} \vec{v} = \vec{0}.$$

Multiply both sides with A^{m-2} to see that $c_1 = 0$. etc.

Section 3.4 Coordinates

2) **Solution 1.** Work entirely inside the plane.

Solve the system of equations $x\vec{v}_1 + y\vec{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$ which leads to $x = 3, y = 1$. In the plane,

the coordinates of the vector \vec{x} is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution 2. Work in space.

The vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is perpendicular to the plane so that we can take

$$S = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

as the coordinate transformation. The inverse is

$$S^{-1} = \begin{bmatrix} -1 & -2 & 5 \\ -2 & 2 & -2 \\ 1 & 2 & 1 \end{bmatrix} / 6$$

and $[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

$$14) A = \begin{bmatrix} 7 & -1 \\ -6 & 8 \end{bmatrix}, S = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

The matrix in the coordinates of the new basis is $B = S^{-1}AS = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

16) a) Note that the two first basis vectors are in the plane while the third is perpendicular to the plane. Therefore, in that basis, the transformation is given by the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

The matrix in the standard basis is $A = SBS^{-1}$, where $S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}$. We get

$$A = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix} / 7.$$

22) The vector \vec{x} is $-\vec{v} + 2\vec{w}$. Flip the vector \vec{v} and add two times the vector \vec{w} in the picture.

26) Yes, the transformation is linear because it is given by a matrix S^{-1} .

32)* a) The hint gives the solution away see 56) above.

b) $\vec{v}_1 = A^2\vec{v}$ is mapped to $A^3\vec{v} = 0$, $\vec{v}_2 = A\vec{v}$ is mapped to $A^2\vec{v} = \vec{v}_1$ and $\vec{v}_3 = \vec{v}$ is mapped to $A\vec{v} = \vec{v}_2$. Therefore, the matrix in that basis is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

38)* Yes, they are similar. We will learn later a general method to check such things. At this stage of the course, we have maybe to experiment a bit. For example:

1) $S_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ flips both the diagonal elements as well as the side diagonal elements.

2) $S_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ flips the side diagonals as well as the signs of the side diagonals.

3) $S_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ changes the signs of the side diagonals.

Bingo! Combining 2) and 3) achieves the goal $S = S_1S_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ just flips the side diagonals and so

$$\begin{bmatrix} a & d \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Note that $S = S^{-1}$ in this case.