

## Section 2.4 Matrix Multiplication.

$$4) \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$$

$$14) BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}, BD = [6], CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, EB =$$

$$[5, 10, 15]. \text{ If you want to pair matrices with itself, we can also form } A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} C^2 =$$

$$\begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}, E^2 = [25].$$

$$28) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ is a possibility.}$$

Remark for later: Actually,  $A^2 = 0$  implies that  $A$  is similar to that transformation (see later). The image of  $A$  is contained in the kernel of  $A$  and must be one dimensional by the dimension formula (seen later). In a basis, where one of the basis vectors is in the kernel, the transformation looks like that.

$$40) AB = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}. A = (AB)B^{-1} = \begin{bmatrix} 4 & 5 \\ -1 & -1 \end{bmatrix}.$$

$$76) \text{ a) } A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$$

$$\text{b) } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\text{c) } AB = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ -1 & 0 & 1/3 \end{bmatrix}.$$

$$\text{d) } M = A^{-1}BA = \begin{bmatrix} 2/3 & 1/3 & -2/9 \\ 0 & 1 & 0 \\ -1 & 0 & 1/3 \end{bmatrix}.$$

$$48) T : P_2 \rightarrow P_2, P_1 \rightarrow P_3 \rightarrow P_0 \rightarrow P_1.$$

$$L : P_0 \rightarrow P_0, P_3 \rightarrow P_3, P_1 \rightarrow P_2 \rightarrow P_1.$$

a) Inverse rotation  $T^{-1}$ . b) Reflection

c) Since  $T^3 = I$ , we know that  $T^2$  is the inverse rotation  $T^{-1}$ .

d)  $T \circ L : P_0 \rightarrow P_1, P_1 \rightarrow P_2, P_2 \rightarrow P_3, P_3 \rightarrow P_0$ .

$L \circ T : P_0 \rightarrow P_2, P_1 \rightarrow P_3, P_2 \rightarrow P_1, P_3 \rightarrow P_0$ .

e)  $P_0 \rightarrow P_2 \rightarrow P_3 \rightarrow P_0, P_1 \rightarrow P_1$  is a rotation.

$$66) A \text{ is invertible if and only if } A_{11} \text{ and } A_{22} \text{ are both invertible. The inverse is } \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}.$$

One can find the inverse by writing out  $BA = 1$  as four equations for the four unknown entries of  $B$ .

### Section 3.1 Image and Kernel

10) We bring the matrix in row reduced echelon form. Starting to clean out the second column  $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  and end up with  $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ . If we write down the

system  $B \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ , we get  $w = 0, z = s, y + 2s = 0, x - s = 0$ , where  $s$  is a free variable introduced in the third column, the only one without leading one. The kernel is the line

spanned by the vector  $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ .

22)  $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$  and  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ . The image is spanned by  $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ ,

the two first columns of  $A$ . The kernel is spanned by  $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ .

34) Take the projection on the plane  $-x + y + 2z = 0$  for example. There are other possibilities.

44) Under a row reduction, both the image and kernel change.

Both in a) and b) we can take as a counter example the matrix  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . The kernel of  $A$

is spanned by  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  the image of  $A$  is spanned by  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The kernel of  $\text{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  is

spanned by  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , the image of  $\text{rref}(A)$  is spanned by  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

54) If there was no error, then  $Hw = 0$  because  $H(Mx) = 0$  if there was no error. The columns of  $H$  encode, which of the bits are switched. We can look match  $H(M(x + e))$  with the columns and get the offending bit.

38) a)  $Ax = 0$  implies  $A^2x = 0$  so that  $\ker(A^2)$  contains  $\ker(A)$  and more generally,  $\ker(A^n)$  contains  $\ker(A^{n-1})$ . The inclusion can be strict, like with  $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ .

b)  $\text{im}(A^2)$  is a subset of  $\text{im}(A)$  and more generally  $\text{im}(A^n)$  is a subset of  $\text{im}(A^{n-1})$ . The inclusion can be strict, like with  $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$  where the image of  $A^2$  is contained in the one dimensional  $\text{im}(A)$ .

48)

a)  $w = Av$  and  $Aw = A^2v = Av = w$  so that  $Aw = w$ .

b) If  $\text{rank}(A) = 2$  then  $A^2 = A$  implies  $A = 1$ .

If  $\text{rank}(A) = 0$ , then  $A = 0$ .

c) If  $\text{rank}(A) = 1$ , then  $Ax$  is a multiple of a vector  $w$ . Assume  $Au = 0$ . Since by a),  $Aw = w$ , and  $Au = 0$ , the two vectors  $w$  and  $u$  span the plane. A general vector is of the form  $su + tw$ .

We have  $Ax = tw$  so that  $A$  is a projection along  $u$  onto the line spanned by  $w$ .

### Section 3.2 Basis

6) The intersection is a linear subspace, the union not.

18) Linear dependent.

24) Finding the basis of the orthogonal complement is identical to find a basis for the kernel of

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

We can take  $\begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$  for example.

28) The columns are linearly independent and form a basis.

48) It is the kernel of the matrix  $A = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ . It is the image of a matrix  $B = \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$ .

36) The image vectors are linearly dependent too.

38)

a) See fact 3.2.6.

b) If they would not span  $V$ , we could add an other vector and still have a linear independent set.

c) Just take the vectors  $v_j$  as the columns of the matrix.