Section 2.1 Linear transformations. (6,14,22,28,42,44,34*,(24-30)*)

6) Row picture: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are the rows of the matrix, we are looking for. With $A == \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ one has $A\vec{x} = T(\vec{x})$. Indeed, the transformation is linear.

14) a) In order to see whether a matrix is invertible, we row reduce.

$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 15-2k \end{bmatrix}$. The matrix is not invertible, if and only if $2k = 15$.

b) All entries are integers if the solution $\vec{x} = (x, y)$ to the equation $A\vec{x} = \vec{b}$ with integer vector $\vec{b} = (a, b)$ has integers. If we row reduce the augmented matrix $B = \begin{bmatrix} 2 & 3 & a \\ 5 & k & b \end{bmatrix}$ we end up with
$\text{rref}(B) = \begin{bmatrix} 1 & 0 & (3b-ak)/(2k-15) \\ 0 & 1 & (5a-2b)/(2k-15) \end{bmatrix}$ which shows that $2k - 15$ must be either a fraction $1/n$. That means $k = (15 + 1/n)/2$, where n is a nonzero integer.

22) Because the y coordinate changes sign, this is a reflection at the x axes.

28) The face gets streched in the y coordinates by a factor 2.

42) a) The image of $\vec{0}$ is $\vec{0}$. The image of the 3 basis vectors e_i are the columns of A. The image of the other 4 vectors can be obtained by linearity.

b) To find the points which are maped to zero, we must have $-x/2 + y = 0$ and $-x/2 + z = 0$. Choosing y freely, say s, we have $x = 2s, y = z = s$. So, all the points on the line $\begin{bmatrix} 2s \\ s \\ s \end{bmatrix}$ are maped to zero.

P.S. We will later call the set of all this vectors the **kernel** of A.

44) Yes, the transformation is linear. The matrix is obtained by applying the map to the fundamental vectors e_i:

$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

34*) The trick is to draw the images of the basis vectors and put that as the columns of the matrix $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

24) is a rotation by 90 degrees in the counter clockwise direction,

25) is a scaling (dilation) by a factor 2,

26) is a reflection at the line $x = y$,

27 is a reflection at the y axes,

28) streches by a factor 2 in the y direction.

29) is a reflection at the origin,

30) is a projection onto the y-axes.

Section 2.2 Linear transaformations. (2.2 4,8,10,30,34,47*,50* 10/3)

4) It is a rotation dilation. A dilation by a factor $\sqrt{2}$ followed by a rotation by $\pi/2$.

8) A shear parallel to the x axes.

10) Just take the basis vectors and map them under the map $T(\vec{x}) = (\vec{u} \cdot \vec{x})\vec{u}/(25)$. The matrix is $\begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$.

30) It is a projection on the line containing the vector $(1, 2)$.

34) For the transformation which is a projection onto a line, all three basis vectors e_i should be maped into the same vector. That is the case for B only. For a reflection in a line, the map should have the property that $T(T(x) = x$. This is only the case for transformation E.

50) Assume, we have two perpendicular vectors $\vec{v}_1$, $\vec{v}_2$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are orthogonal. The unit circle can be written as $cos(t)v_1 + \sin(t)v_2$. The image is $\cos(t)w_1 + \cos(t)w_2$.

An other approach would be to verify that transformations like rotations, dilations, diagonal transformations and reflections map ellipses into ellipses and that one can write any transoformation as a composition of such transformations.

Section 2.3 Linear transformations. (2.3 10,20,30,38,40,26*,42*)

10)

Doing Gauss-Jordan reduction on $\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 3 & | & 0 & 1 & 0 \\ 1 & 3 & 6 & | & 0 & 0 & 1 \end{bmatrix}$. gives $\begin{bmatrix} 1 & 0 & 0 & | & 3 & -3 & 1 \\ 0 & 1 & 0 & | & -3 & 5 & 2 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{bmatrix}$.

20) Doing Gauss-Jordan reduction on $\begin{bmatrix} 1 & 3 & 3 & | & 1 & 0 & 0 \\ 1 & 4 & 8 & | & 0 & 1 & 0 \\ 2 & 7 & 12 & | & 0 & 0 & 1 \end{bmatrix}$, we end up with

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & -15 & 12 \\ 0 & 1 & 0 & 4 & 6 & -5 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right].$$

30) The map is never invertible.

38) Apply the formula:

$$\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}.$$

40) If the i'th and j'th row are the same, then the vector $\vec{u} = e_i - e_j$ is maped to the zero vector This means that the vectors $\vec{v}$ and $\vec{v} + \vec{u}$ are maped into the same point and the inverse of that vector is not defined.

26) The map is invertible. Given y_1, y_2 we immediatly get $x_2 = y_1$, then $x_1^3 = y_2 - y_1$ and so $x_1 = (y_2 - y_1)^{1/3}$.

42) Yes, permutation matrices are invertible, because we can invert the permutation. For example, the inverse of the permutation matrix which belongs to the permutation (1,2,3) -¿ (2,3,1) is the permutation matrix which belongs to the permutation (1,2,3) -¿ (3,1,2).