Solutions to 1. Week Homework Math 21b

If you have questions about posted solutions (like typos, errors, additions, things needing clarification), please email it to math21b@fas.harvard.edu.

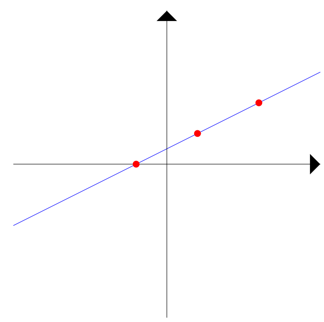
Section 10.1 Linear equations.
10) The equation are

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 4y + 7z &= 2 \\ 3x + 7y + 11z &= 8 \end{aligned}$$

Subtracting twice 1) from 2) gives $z = 0$. Subtracting 3 times 1) from 3) gives $y = 5$, then $x = -9$.

12) The solution is the intersection of two lines

$$\begin{aligned} x - 2y &= 3 \\ 2x - 4y &= 6 \end{aligned}$$

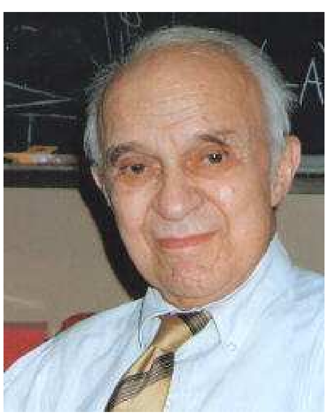


But these two lines are the same.

20) The Leontief equations are

$$\begin{aligned} a - 0.1b &= 1000 \\ b - 0.2a &= 780 \end{aligned}$$

This system can be solved by adding 5 times the second equation to the first: $4.9b = 4900$ or $b = 1000$ and $a = 1100$.



Remark. Wassily Leontief was born in 1906 in St. Petersburg Russia. He obtained his doctorat in Germany in 1928, was Harvard professor from 1953 to 1975. He received in 1973 the Nobel prize in Economics. He worked last at New York University and died in 1999. He is best known for the development of "input-output" analysis, which has fundamentally influenced economic analysis. In this theory, the structure of the economy is described by a matrix of input-output coefficients which turned out to be very useful: Leontief: "I was not trying to improve the system. I was just concentrating on understanding how it works." (Source: J.S, Landefeld and S.H. McCulla: "Survey of current business", March 1999, p. 9-11).

24) Assume x is the speed of the river and y is the speed of the ship. If L is the distance to travel, then $8000 = 20(x+y)$ and $8000 = 40(x-y)$. We get the system

$$\begin{aligned} 400 &= x+y \\ 200 &= x-y \end{aligned}$$

Addition of the two lines gives $2x = 600$ or $x = 300$ (meter per minutes) and $y = 100$ (meter per minutes).

32b) The last equation gives $x_4 = 0$. The second last $x_3 = 2$, the second, $x_2 = -1$, the first finally $x_1 = 1$.

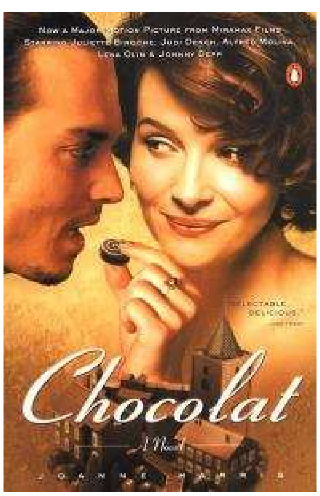
26*) Simplify the system

$$\begin{aligned} x+y-z &= 2 \\ x+2y+z &= 3 \\ x+y+(k^2-5)z &= k \end{aligned}$$

to get

$$\begin{aligned} y+2z &= 2 \\ (k^2-4)z &= k-2 \end{aligned}$$

so that $z = (k-1)/(k^2-4)$. For $k \neq 2, -1$ we have exactly one solution. For $k = 2$, the variable z can be anything in the last equation and we have infinitely many solutions. For $k = -2$, the last equation has no solution z. The system is then inconsistent.



36*) If Boris fortune is x and Marinas y, then

$$\begin{aligned} x/2+y &= 2 \\ x+y/2 &= 1 \end{aligned}$$

Subtracting twice the second from the first equation gives $3/2x = 0$ and $y = 2$.

Section 10.2 Gauss-Jordan Elimination

6) The system is already in row reduced echelon form. We can chose $x_5 = c$ freely, Then $x_4 = 1 - c$, $x_3 = 2 + 2c$, $x_2 = d$ can be chosen freely and we have $x_1 = 3 - c + 7d$. We see that the solution set has two free parameters.

12) After doing the Gauss-Jordan elimination, you should end up with the row reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 7/2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 & 1 \end{bmatrix} .$$

18) Just b) and d) are in row reduced echelon form. In a), we can continue with the third row. In c), the third row should be exchanged with the second.

20) There are 4 possible cases:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where c is a constant.

30) In order that the points are on the cubic curve, four equations have to be satisfied:

$$\left| \begin{array}{l|l} (0,1) & a = 1 \\ (1,0) & a + b + c + d = 0 \\ (-1,0) & a - b + c - d = 0 \\ (2,-15 & a + 2b + 4c + 8d = -15 \end{array} \right|$$

To do Gauss-Jordan elimination, we start with

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 2 & 4 & 8 & -15 \end{bmatrix}$$

and end up with

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Therefore, $d = -2, c = -1, b = 2, a = 1$. The cubic polynomial is $1 + 2x - x^2 - 2x^3$.

32) The condition $f_i(a_i) = f_{i+1}(a_{i+1})$ assures that the curves meet at the end points. The condition $f_i'(a_i) = f_{i+1}'(a_{i+1})$ assures that the joint curve does not have corners. The condition $f_i''(a_i) = f_{i+1}''(a_{i+1})$ assures even a better fit in that the curvatures of the curves at the end points agree.

Because we have n curves where each curve has 3 parameters, there are $3n$ variables. There are also 3n equations.

38) The demand vectors are

$$v_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.2 \end{bmatrix} v_2 = \begin{bmatrix} 0.2 \\ 0 \\ 0.5 \end{bmatrix} v_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0 \end{bmatrix}$$

The equation $\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + \vec{b}$ tells that the output vector is the the sum of the consumer demand and suitably scaled demands from the other industries.

Section 10.3 On Solutions of Linear Equations.

4) In order to compute the rank, we do Gauss-Jordan elimination:

$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ The rank is the number of leading ones, which is 2 in this case.

14) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+4+3 \\ -2+4+4 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. In both cases, we get as a result $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

34) This is an important thing to remember: for any $m \times 3$ matrix, Ae_1 is the first column vector, Ae_2, the second and Ae_3 is the third column vector.

a) $A(\vec{x} + \vec{x}_h) = A\vec{x} + A\vec{x}_h = A\vec{x} + 0 = A\vec{x}$.
b) $A(\vec{x} - \vec{y}) = A\vec{x} - A\vec{y} = \vec{b} - \vec{b} = 0$.
c) It is a line parallel to the other line passing through the end point of $\vec{x}_1$.

50) There are no solutions because we have a leading one at the end of the last row which would mean $0x + 0y + 0z = 1$.

26*) It can have a unique solution or no solution.

Assume that the system $Ax = b$ is in row reduced echelon form say $A^*x^* = b^*$ and that $B = [A|c]$ goes after row reduction into $B^* = [A^*|c^*]$. There is one solution if and only if the last entry of c^* is zero.

46*) The rank is three. Row reduction produces the identity matrix.