11/5/2019 Second Hourly Practice 3 **Math 21a, Fall 2019**

Name:

MWF 9 Oliver Knill
MWF 9 Arnav Tripathy
MWF 9 Tina Torkaman
MWF 10:30 Jameel Al-Aidroos
MWF 10:30 Karl Winsor
MWF 10:30 Drew Zemke
MWF 12 Stepan Paul
MWF 12 Hunter Spink
MWF 12 Nathan Yang
MWF 1:30 Fabian Gundlach
MWF 1:30 Flor Orosz-Hunziker
MWF 3 Waqar Ali-Shah

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F

The point $(-5, 3)$ is a critical point of $f(x, y) = 3x^2 + 5y^2$.

Solution:
The function f has only $(0, 0)$ as a critical point.

2) T F

If a function $f(x, y, z)$ has gradient satisfying $|\nabla f| = 1$ everywhere, then the level surface $f(x, y, z) = 1$ is a sphere.

Solution:
Its no problem to get a plane for example with this property.

3) T F

The chain rule assures that the vector $\nabla f(\vec{r}(t))$ and the velocity vector $\vec{r}\,'(t)$ for any curve $\vec{r}(t)$ on the level surface are perpendicular.

Solution:
Yes, this is the gradient theorem.

4) T F

The function $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$ satisfies the formula $D_{\vec{d}}D = \nabla D \cdot \vec{d}$ for any unit vector $\vec{d}$.

Solution:
This is true for all functions f, not only for the D.

5) T F

The function $f(x, y) = x^4 - 1$ has infinitely many critical points.

Solution:
All points on $x = 0$ are critical points.

6) T F

The points $(0, 1)$ and $(0, -1)$ are maxima of $f(x, y) = y^2$ under the constraint $g(x, y) = x^2 + y^2 = 1$.

Solution:
Yes, by inspection

7) T F The function $f(x,y) = y^2$ satisfies the partial differential equation $u_{yx}(x,y) = u_x(x,y)$.

Solution:
Just differentiate.

8) T F Let $f(x,y) = x^3 y$. At every point (x,y) there is a direction $\vec{v}$ for which $D_{\vec{v}} f(x,y) = 0$.

Solution:
It is a general fact for directional derivatives as $D_{\vec{v}} f(x,y) = -D_{\vec{v}} f(x,y)$ implies that if the directional derivative is positive in one direction, it is negative in the opposite direction.

9) T F If $f_{xy} = f_{yx}$ then $f(x,y) = xy$.

Solution:
While Clairaut's theorem is always true, the second statement is not necessary. We can have for example $f(x,y) = \sin(xy) + y^5$ and Clairaut still holds.

10) T F $g(x,y) = \int_0^x \int_0^y f(s,t) \; dtds$ satisfies the partial differential equation $g_{xy}(x,y) = f(x,y)$.

Solution:
By the fundamental theorem of calculus we have $g_{xy} = f(x,y)$.

11) T F If $f(x,y) = g(x,y) = x^2 + y^4$, then the Lagrange problem for maximizing f under the constraint $g(x,y) = 1$ has infinitely many solutions.

Solution:
Yes, every point on the constraint is a maximum.

12) T F The number $|\nabla f(0,0)|$ is the maximal directional derivative $|D_{\vec{v}} f(0,0)|$ among all unit vectors $\vec{v}$.

Solution:
By the cos-formula.

13) | T | F |

Any continuous function $f(x, y)$ takes a global maximum as well as a global minimum on the region $0 \leq x^2 + y^2 \leq 1$.

Solution:
This is the Bolzano extremal value theorm.

14) | T | F |

For any continuous function, $\int_0^1 \int_0^1 f(r, \theta) r \; dr d\theta = \int_0^1 \int_0^1 f(x, y) \; dx dy$.

Solution:
The domain is not correctly integrated

15) | T | F |

If the Lagrange multiplier λ at a solution to a Lagrange problem is positive then this point is a minimum.

Solution:
The sign of λ has nothing to do with the nature of the critical point.

16) | T | F |

The equation $f_{xy} f_{xx} f_{yy} = 1$ is an example of a partial differential equation.

Solution:
Yes, it is an equation for a function f involving partial derivatives.

17) | T | F |

If the discriminant D appearing in the second derivative test of $f(x, y)$ is positive at $(0, 0)$ then $|\nabla f(0, 0)| > 0$.

Solution:
Take a minimum of $f(x, y) = x^2 + y^2$.

18) | T | F |

If $f(x, y)$ is a continuous function then $\int_7^9 \int_5^7 f(x, y) \quad dx dy = \int_5^7 \int_7^9 f(x, y) \; dx dy$.

Solution:
Take a simple example like $f(x, y) = x$, then $f(y, x) = y$, which gives an other result.

19) T F If f has the critical point $(0,0)$, then $f_y + f_x$ has the critical point $(0,0)$.

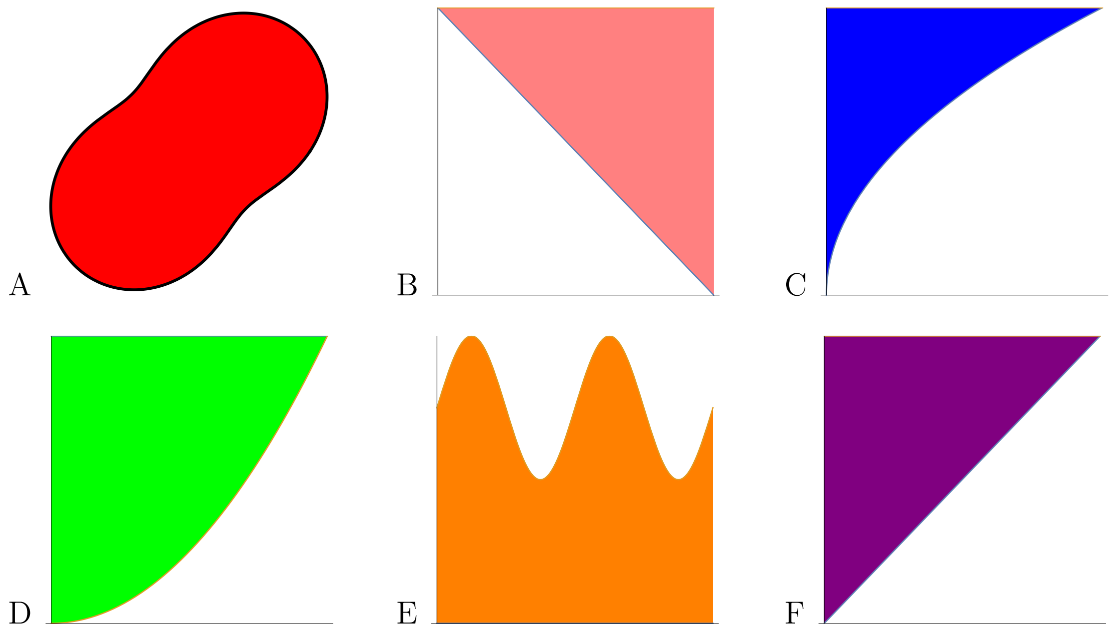
Solution:
Take an example $xy + y^2$

20) T F If $f(x,y)$ takes arbitrary large values, then $g(x,y) = |\nabla f(x,y)|$ takes arbitrary large values.

Solution:
A function can be unbounded without its derivative being unbounded. This already holds in single variable clalculus. A concrete counter example is $f(x,y) = x + y$.

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A - F$.



Enter A-F	Integral
	$\int_0^{2\pi} \int_0^{y} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{\sqrt{y}} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_{2\pi-x}^{2\pi} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{y^2} f(x,y)\ dxdy$
	$\int_0^{2\pi} \int_0^{3+\sin(2x)} f(x,y)\ dydx$
	$\int_0^{2\pi} \int_0^{3+\sin(2t)} f(r,t)\ rdrdt$

b) (4 points) We define the **complexity** of a partial differential equation for $u(t,x)$ or $u(x,y)$ as the number of derivatives appearing in total. For example, the partial differential equation $u_{xxx} = u_{tx}$ has complexity 5 because 5 derivatives have been taken in total. As an expert in PDEs, you know a few of them. Write down the complexities of the partial differential equations. These are integers ≥ 2 in each case.

Complexity	Name
	Laplace for $u(x,y)$
	Wave for $u(t,x)$
	Transport for $u(t,x)$
	Heat for $u(t,x)$


Taylor


Fourier


d'Alembert

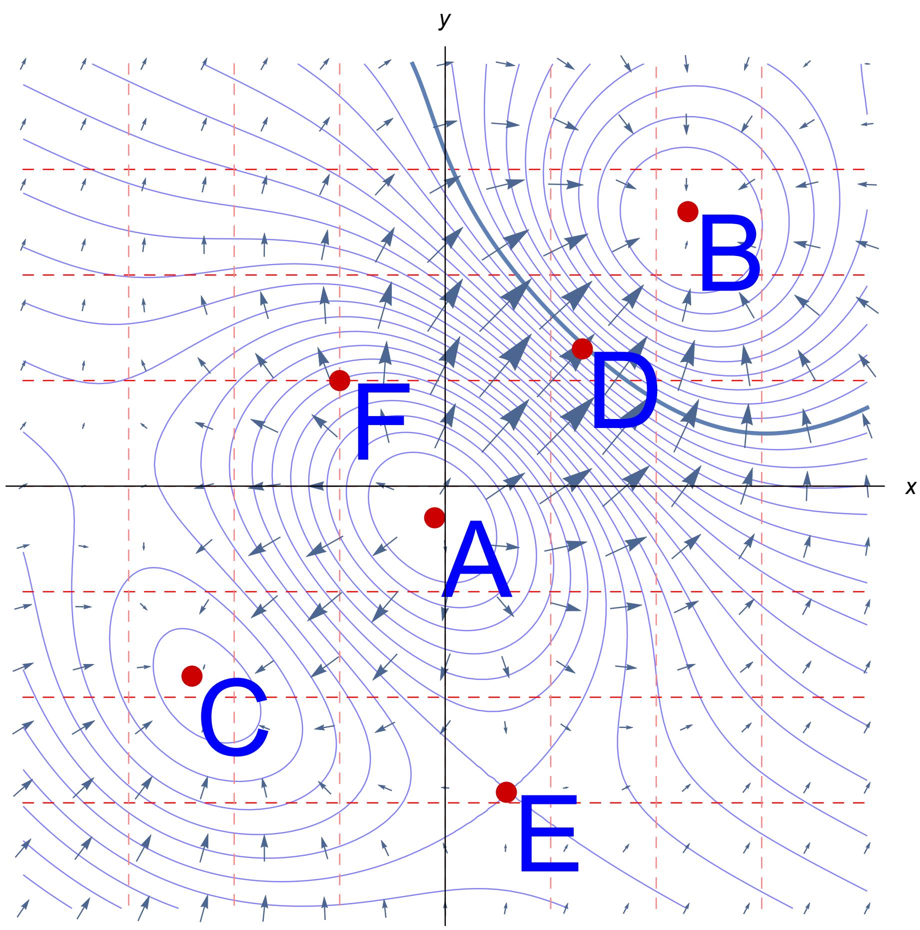

Laplace

Solution:
FDBCEA
4,4,2,3

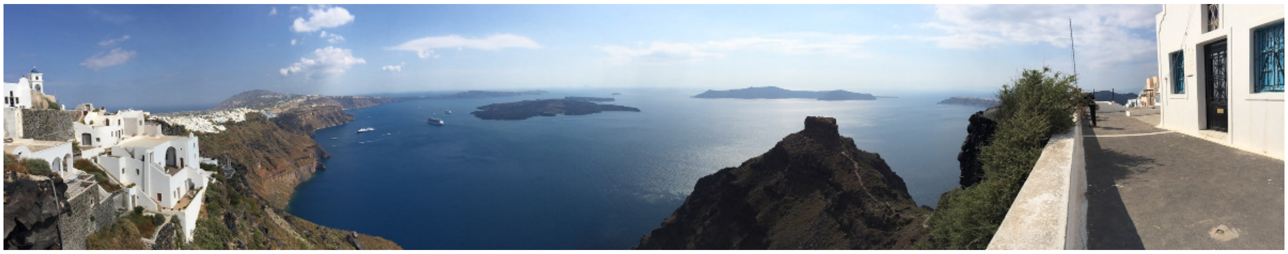
Problem 3) (10 points)

3a) (5 points) In the following contour plot of a height function $f(x, y)$, neighboring contours $f(x, y) = c$ have height distance 1. The arrows indicate the gradient of f. Every point A-F occurs at most once.

Which of the points is the global maximum on the visible region?	
Which of the points is a global minimum on the visible region?	
Which of the points is a global maximum for the function $\lvert\nabla f(x, y)\rvert^2$	
Which of the points is a saddle point?	
Which of the points has the property that $f_x f_y < 0$ at this point?	



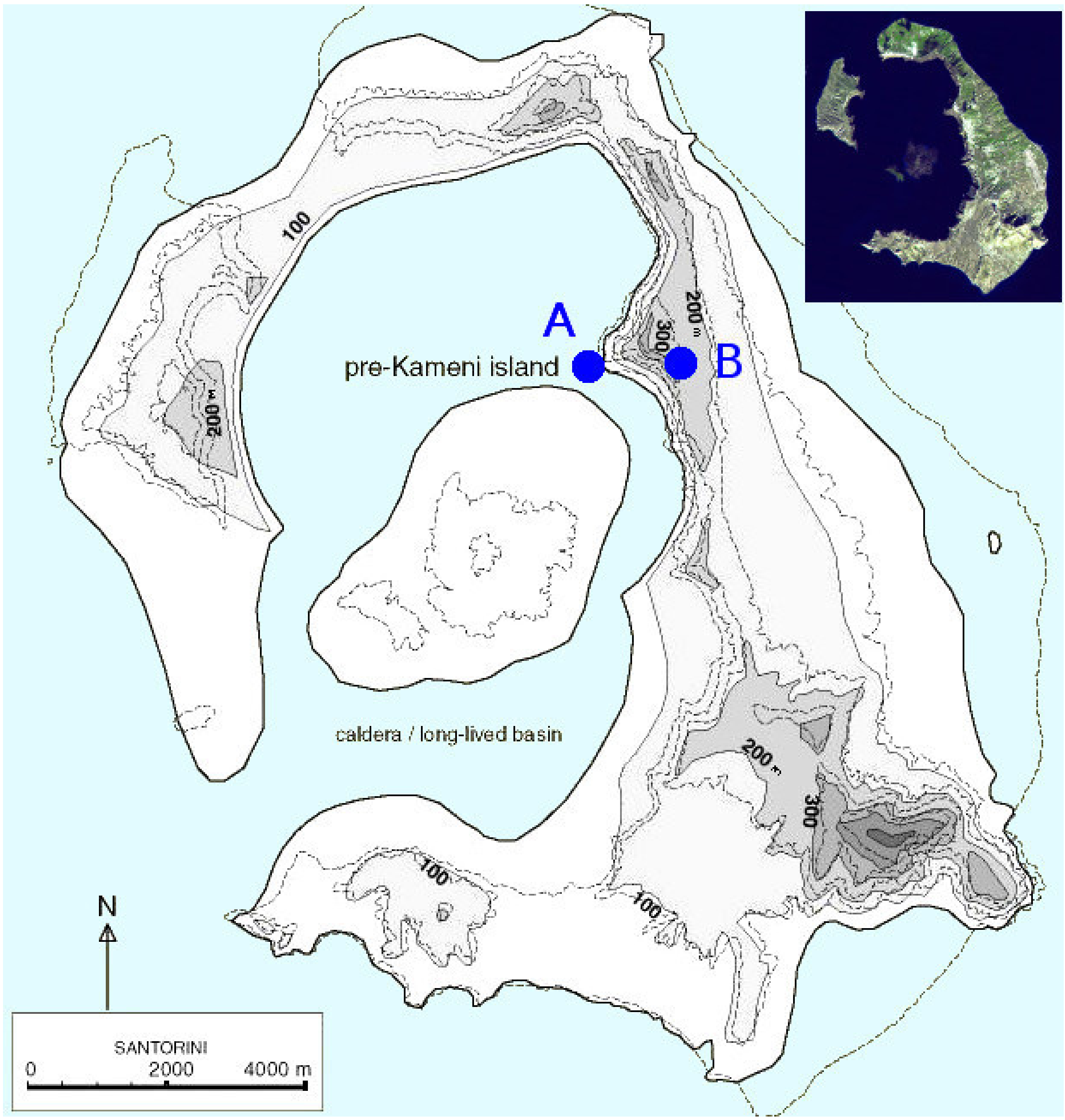
Part b) and c) of the problem are unrelated and on the new page.



Santorini panorama from Imerovigli with view onto Skaros rock, Caldera basin and volcanic island Nea Kameni. Photo: Oliver Knill, June 2015

3b) (2 points) You see a contour map of the Greek island of **Santorini**. Point A is on the water (0 elevation) Point B is **Skaros rock**, which used to be a fortification protecting merchants from pirates. Estimate the average directional derivative between A and B in the direction from A to B. Given elevation markers 100,200,300 are in meters.

Derivative	Check one
2	
0.2	
0.02	



Source: http://www.decadevolcano.net, the picture shows a reconstruction of pre-Minoan Thera done by Druitt and Francaviglia from 1991. The island of today is shown in dottet curves. A satelite picture of the Santorini Caldera with the Nea Kameni volcano in the center is seen in the upper right corner.

3c) (3 points) Which statements about a critical point with discriminant $D \neq 0$ always hold for a smooth function $f(x, y)$?

Critical Point	$f_{xx} > 0$	$f_{yy} < 0$	$f_x > 0$	$f_y < 0$
Maximum				
Minimum				
Saddle point				

Solution:
a) B is the global maximum, A is the global minimum, D is the point with largest steepness $|\nabla f(x,y)|$. The point E is a saddle point. At the point F, the gradient is parallel to $[-1,1]$ showing that $f_x f_y < 0$.
b) The distance is about 1500 m and the height difference about 300 m which gives a value of $\boxed{0.2}$.
c) Since D is not zero, we have $f_{xx} > 0$ at a minimum and $f_{yy} < 0$ at a maximum since any three cases are critical points, the last two columns are empty. At a saddle point, we can have $f_{xx} > 0$ or $f_{yy} < 0$ but it does not need to be. Examples like $x^2 - y^2$ or $y^2 - x^2$ show that either case can occur. So:

Critical Point	$f_{xx} > 0$	$f_{yy} < 0$	$f_x > 0$	$f_y < 0$
Maximum		X		
Minimum	X			
Saddle point				

Problem 4) (10 points)

a) (6 points) Let $g(x,y) = (6y^2-5)^2(x^2+y^2-1)^2$. Find the gradient of g at the points $(1,-1)$, $(-1,1)$ and $(1,1)$.

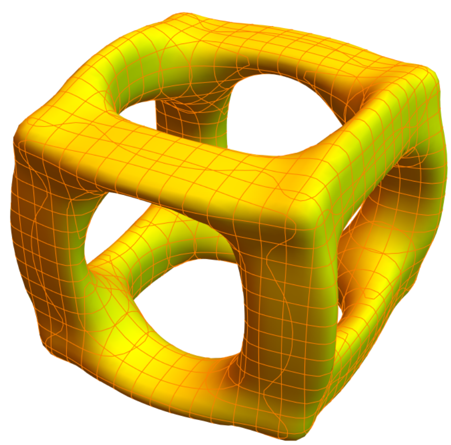
b) (4 points) A student from the **Harvard graduate school of design** contemplates the surface

$$f(x,y,z) = g(x,y) + g(y,z) + g(z,x) = 3$$

shown in the picture. She first discovers the formula

$$\begin{aligned} \nabla f(1,-1,1) \quad = \quad & [g_x(1,-1) + g_y(1,1) \ , \\ & g_x(-1,1) + g_y(1,-1) \ , \\ & g_x(1,1) + g_y(-1,1)] \ . \end{aligned}$$

Without verifying this, find the tangent plane at $(1,-1,1)$.



Solution:
a) Compute the gradient

$$\nabla g(x,y) = [4x(6y^2-5)^2(x^2+y^2-1), 24y(6y^2-5)(x^2+y^2-1)^2+4y(6y^2-5)^2(x^2+y^2-1)] \ .$$

Plugging in the values gives $(4,-28)$, $(-4,28)$ and $(4,28)$.
b) As we have the g_x, g_y values from a), we have just to plug in the numbers and get $\nabla f = [32,-32,32]$. The equation is $32x - 32y + 32z = d$. Now plug in the point $(1,-1,1$ to get $32+32+32 = 96$ and the plane is $\boxed{x - y + z = 3}$.

Problem 5) (10 points)

Octagons are used in architecture designs, in symbolism, for rugs or in traffic signs. Use the Lagrange method to find the octagon with maximal area

$$f(x,y) = (x+2y)^2 - 2y^2$$

if the circumference

$$g(x,y) = 4x + 4y\sqrt{2} = 8 \; .$$

is fixed.

Solution:
Write down the Lagrange equations

$$\begin{aligned} 2(x+2y) &= \lambda 4 \\ 4(x+2y) - 4y &= \lambda 4\sqrt{2} \\ 4x + 4y\sqrt{2} &= 8 \end{aligned}$$

Dividing out λ gives $y = x/\sqrt{2}$. Plugging this into the third equation gives $x = 1$ and $y = 1/\sqrt{2}$.

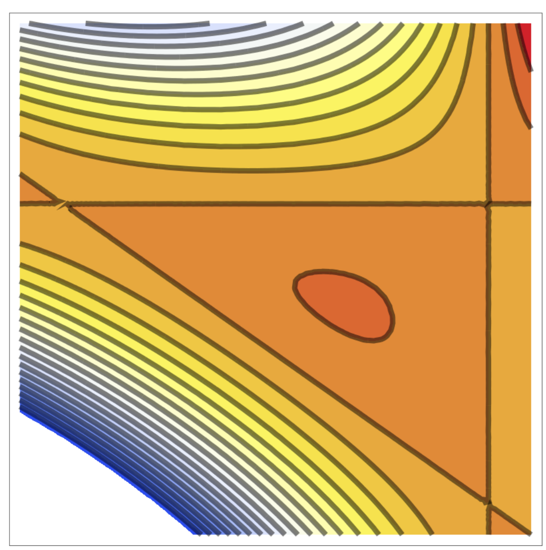
Problem 6) (10 points)

a) (8 points) Find and classify the four critical points of the **"triangle function"**

$$f(x,y) = x^2y + y^2x - y^2 - y$$

using the second derivative test. There is no need to find the values of f.

b) (2 points) State whether any of the four points is a global maximum or minimum on the entire plane.



Solution:
a) Compute the gradient $\nabla f(x,y) = [2xy + y^2, -1 + x^2 - 2y + 2xy]$ and set it to $(0,0)$. Factor out the y in the first $y(2x+y=0)$ shows that either $y=0$ leading to two solutions $x = \pm 1$ or $y = -2x$ which leads to a quadratic equation with $x = 1, x = 1/3$ as solutions. We compute also f_{xx} and D and get the classification

x	y	D	f_{xx}	Type	f
-1	0	-4	0	saddle	0
1/3	-2/3	4/3	-4/3	maximum	8/27
1	-2	-4	-4	saddle	0
1	0	-4	0	saddle	0

b) There is $\boxed{\text{no global maximum}}$ as $f(x,x) = 2x^3 - x^2 - x$ has no global maximum or minimum.

Problem 7) (10 points)

The region R defined by

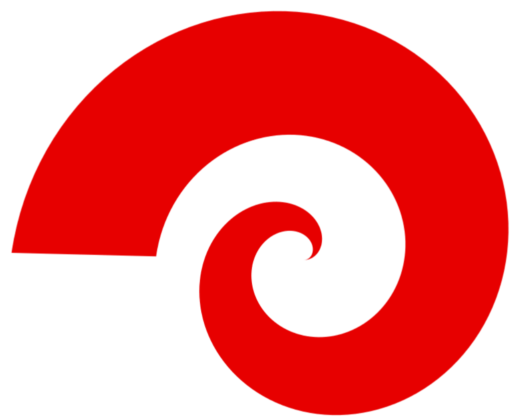
$$\theta \leq r(\theta) \leq 2\theta$$

with

$$0 \leq \theta \leq 3\pi$$

is shown in the picture. Compute its **moment of inertia**

$$\iint_R x^2 + y^2 \, dA \, .$$



Solution:
The integral is

$$\int_0^{3\pi} \int_\theta^{2\theta} r^2 \cdot r \, dr d\theta$$

which is $729\pi^5/4$. As expected, most mistakes came from forgetting the integration factor r or (unexpectedly) that the lower bound θ morphed to 0 for many.

Problem 8) (10 points)

a) (5 points) Find a vector perpendicular to the **tangent line** of the curve

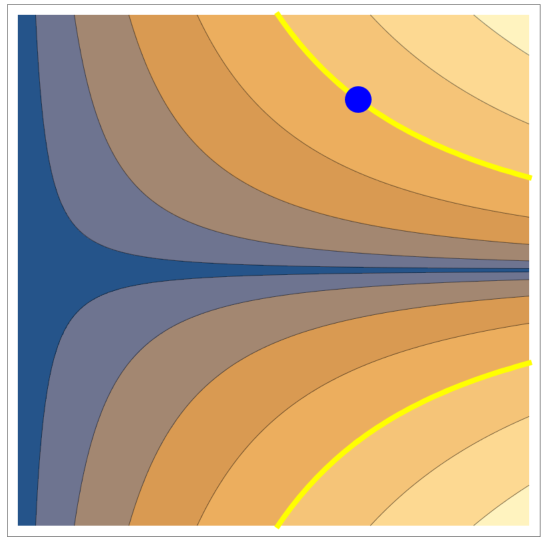
$$f(x,y) = 5(x^3y^2)^{1/5} = 100$$

at $(20,20)$. The picture shows a contour map of f.

b) (5 points) Use the same function in a) to estimate

$$f(21,19) = 5(21^3 \cdot 19^2)^{1/5}$$

by **linearizing** f near $(20,20)$.



Solution:
a) Using the single variable chain rule and multiplication rule correctly to take the derivatives shows that the gradient is $[3,2]$. b) $f_x(20,20) = 3$, $f_y(20,20) = 2$. We estimate $100 + 3 - 2 = 101$. The actual value is 100.88.

Problem 9) (10 points)

We compute the **surface area** of the surface

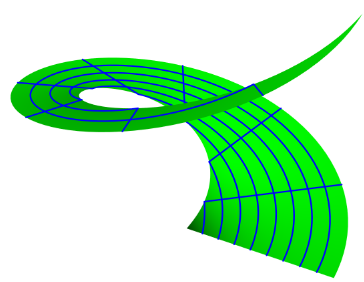
$$\vec{r}(u,v) = [v\cos(u), v\sin(u), u]$$

over the region $R: 0 \leq u \leq 2\pi, u \leq v \leq 2\pi$.

a) (5 points) First verify that the integral is of the form

$$\iint_R \sqrt{1+v^2}\; dudv\ .$$

b) (5 points) Now compute the surface area integral.



Solution:
a) Take the cross product of $\vec{r}_u = [-v\sin(u), v\cos(u), 1]$ and $\vec{r}_v = [\cos(u), \sin(u), 0]$ to get $\vec{r}_u \times \vec{r}_u = [-\sin(u), \cos(u), -v]$ which has length $\sqrt{1+v^2}$.
b) The integral

$$\int_0^{2\pi} \int_u^{2\pi} \sqrt{1+v^2}\, dvdu$$

is not pleasant (doable as you have done in the homework). Better is to switch the order of integration. Draw the region which is a triangle and switch to a (dx dy) integral

$$\int_0^{2\pi} \int_0^{v} \sqrt{1+v^2} dudv = \int_0^{2\pi} v\sqrt{1+v^2} dv = (1/3)(1+v^2)^{3/2}|_0^{2\pi} = ((1+4\pi^2)^{3/2} - 1)/3 \ .$$

The answer is $\boxed{(1+4\pi^2)^{3/2}/3 - 1/3}$.

Problem 9) (10 points)

The **Ramanujan constant** $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925...$ is close to an integer. There is an elaborate story about why this is so. Here, we just want to estimate the logarithm of this constant roughly.

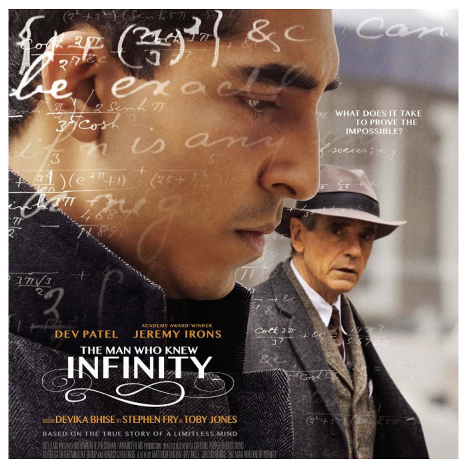
Let

$$f(x,y) = x\sqrt{y} \ .$$

Estimate

$$f(3.141, 163) = 3.141\sqrt{163}$$

near $(x_0, y_0) = (3, 169)$ using linear approximation.



Ramanujan is featured in the movie: "The Man who knew infinity", 2015

Solution:
$\nabla f(x,y) = [\sqrt{y}, x/(2\sqrt{y})]$ which is $[13, 3/(2*13)] = [13, 3/26]$. Now $L(3.141, 163) = 39 + 0.141*13 - 18/(2*13) = \boxed{40.1407}$. The actual result is 40.1016.