11/5/2019 Second Hourly Practice 1 **Math 21a, Fall 2019**

Name:

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MWF 9 Tina Torkaman
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MWF 1:30 Fabian Gundlach
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MWF 3 Waqar Ali-Shah

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) [T] F — The identity $f_{yxyx} = f_{xyxy}$ holds for all smooth functions $f(x, y)$.

Solution:
This is Clairaut

2) T [F] — Using linearization we can estimate $(1.003)^2(1.0001)^4 \approx 2\cdot0.003+4\cdot0.0001$.

Solution:
The constant is missing.

3) [T] F — We have $d/dt(x^2(t)y(t)) = [2x(t)y(t), x^2(t)] \cdot [x'(t), y'(t)]$.

Solution:
This is a direct application of the chain rule.

4) T [F] — The function $f(x, y) = 3y^2 - 2x^3$ takes no maximal value on the "squircle" $x^4 + y^4 = 8$.

Solution:
We can find it using Lagrange. Also Bolzano assures that it exists.

5) T [F] — If $f(x, t)$ solves the heat equation then $f(x, -t)$ solves the heat equation.

Solution:
The sign of f_t changes sign while f_{xx} does not.

6) [T] F — If $f(x, t)$ solves the wave equation, then $f(x, -t)$ solves the wave equation.

Solution:
Both the sign of f_{tt} and f_{xx} change sign.

7) T F — There exists a smooth function on the region $x^2 + y^2 < 1$ so that it has exactly two local minima and no other critical points.

Solution:
You have worked on that in a homework.

8) T F — For a function $f(x, y)$, the vector $[f_x(0,0), f_y(0,0), -1]$ is perpendicular to the graph $f(x, y) = z$ at $(0, 0, f(0, 0))$.

Solution:
Write $g(x, y, z) = f(x, y) - z$ and find the gradient.

9) T F — If a function $f(x, y)$ is equal to its linearization $L(x, y)$ at some point, then $f_{xx}(x, y) = f_{yy}(x, y)$ at every point.

Solution:
Both sides are zero.

10) T F — The equation $f(x, y) = 9x - 5x^2 - y^2 = -9$ implicitly defines $y(x)$ near $(0, 3)$ and $y'(0) = f_x(0, 3)/f_y(0, 3)$.

Solution:
The sign is off

11) T F — If a tangent plane to a surface S intersects S at infinitely many points, then S must be a plane.

Solution:
Take the surface $\sin(x + y)$ for example and the tangent plane $z = 1$ at the point $(\pi/2, 0)$

12) T F — If $\vec{u} = [1, 0, 0]$ and $\vec{v} = [0, 1, 0]$ then $(D_{\vec{u}} D_{\vec{v}} - D_{\vec{v}} D_{\vec{u}})f = D_{\vec{u}\times\vec{v}} f$.

Solution:
The left hand side is zero. The right hand side not necessarily.

13) T **F** The surface area of the parametrized surface $\vec{r}(r,\theta) = [r\cos(\theta), r\sin(\theta), r]$, with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ is $\int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta| r dr d\theta$.

Solution:
There is an r too much.

14) T **F** Let D be the unit disk $x^2 + y^2 \leq 1$. Any function $f(x,y)$ which satisfies $|\iint_D f(x,y)\,dA| = \iint_D |f(x,y)|\,dA$ must have $f(x,y) \geq 0$ on D.

Solution:
Take a function $f = -1$ for example.

15) **T** F The iterated integral $\int_{-1}^{1}\int_{10}^{20} e^{x^2} y^{11}\,dx\,dy$ is zero.

Solution:
Symmetry

16) T **F** The tangent plane to the graph of $f(x,y) = xy$ at $(2,3,6)$ is given by $6 + 3(x-2) + 3(y-3) = 0$.

Solution:
There should also be a z-part.

17) T **F** If the gradient of $f(x,y)$ at $(1,2)$ is zero, then $f(1,2)$ must be either a local minimum or maximum value of $f(x,y)$ at $(1,2)$.

Solution:
It could be a saddle point

18) **T** F If $\vec{r}(t)$ is a parametrization of the level curve $f(x,y) = 5$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Yes because $\vec{r}'$ is perpendicular to the gradient.

19) T F (F marked)

The function $f(x,y) = (x^3+y^3)/(x^2+y^2)^2$ has a limiting value at $(0,0)$ so that it is continuous everywhere.

Solution:
Use polar coordinates to see this.

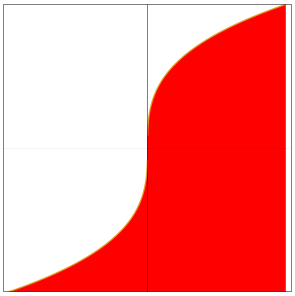
20) T F (T marked)

If the contour curves $f(x,y) = 1$ and $g(x,y) = 1$ have a common tangent line at $(1,2)$ and $|\nabla f(1,2)| = 1 = |\nabla g(1,2)| = 1$, then $(1,2)$ is a solution to the Lagrange equations for extremizing f under the constraint $g = 1$.

Solution:
The gradients are parallel. That is what the Lagrange equations tell.

Problem 2) (10 points) No justifications needed

a) (6 points) Please match each picture below with the double integral that computes the area of the region:

A


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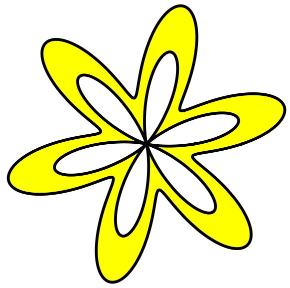
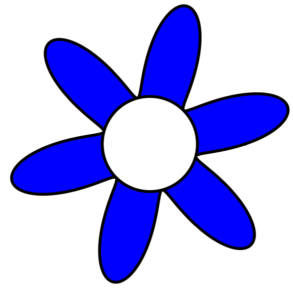
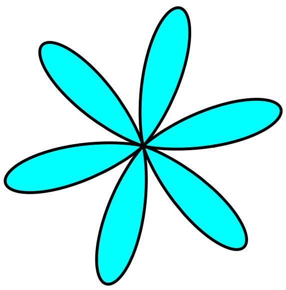
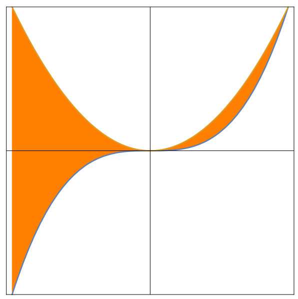
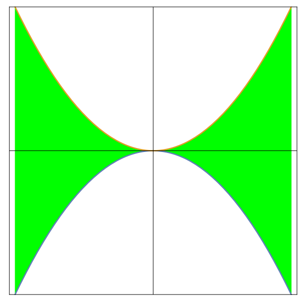


C


D


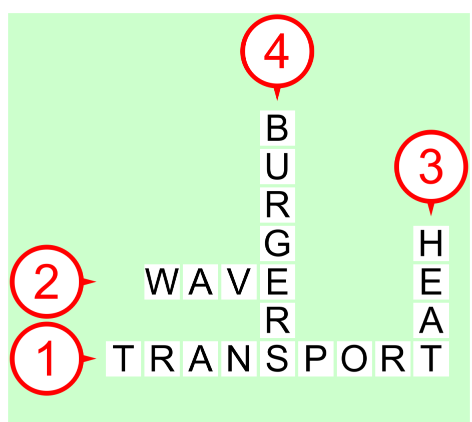
E


F


Enter A-F	Integral
	$\int_0^{2\pi} \int_{1+\sin(6\theta)}^{2+\sin(6\theta)} r \; dr d\theta$
	$\int_{-1}^{1} \int_{-x^2}^{x^2} 1 \; dy dx$
	$\int_0^{2\pi} \int_0^{1+\sin(6\theta)} r \; dr d\theta$
	$\int_{-1}^{1} \int_{y^3}^{1} 1 \; dx dy$
	$\int_{-1}^{1} \int_{x^3}^{x^2} 1 \; dy dx$
	$\int_0^{2\pi} \int_1^{2+\sin(6\theta)} r \; dr d\theta$

b) (4 points) We design a crossword puzzle. Match the PDEs:

Enter 1-4	
	$u_t = u_x$
	$u_t = u_{xx}$
	$u_{tt} = u_{xx}$
	$u_t + uu_x = u_{xx}$



Solution:
a) BFDAEC
b) 1324

Problem 3) (10 points)

3a) (7 points) Fill in the points A-G. There is an exact match. You see the level curves of a function $f(x,y)$ inspired from one of your homework submissions. The circular curve is $g(x,y) = x^2 + y^2 = 1$.

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a) At the point [ ], the function f is a global maximum on $g = 1$.

b) At the point [ ], the function f is a global minimum on $g = 1$.

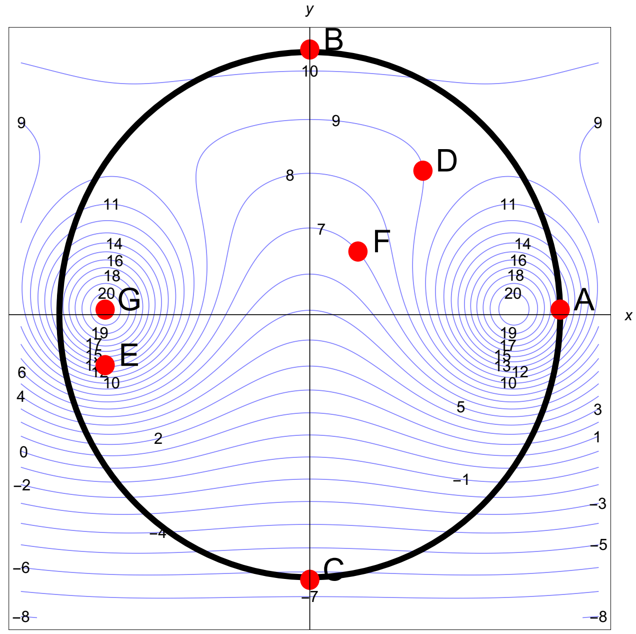
c) At the point [ ], $|f_y|$ is maximal among all points A-G.

d) At the point [ ], $f_x > 0$ and $f_y = 0$.

e) At the point [ ], $f_x > 0$ and $f_y > 0$.

f) At the point [ ], $\nabla f = \lambda \nabla g$ and $g = 1$ for some $\lambda > 0$.

g) At the point [ ], $|\nabla f|$ is minimal among all points A-G.



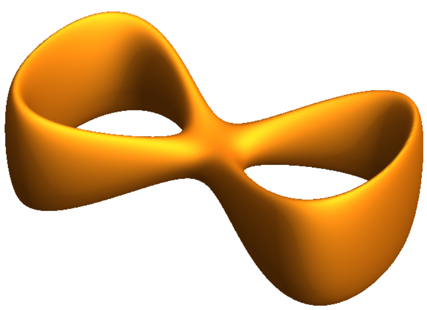
3b) (3 points) Fill in the numbers $1, -1$, or 0. In all cases, the vector $\vec{v}$ is a general unit vector.

a) At a maximum point of $f(x,y)$, we have	$D_{\vec{v}} f =$	
b) At any point (x,y), we have	$\|D_{\vec{v}} f\|/\|\nabla f\| \leq$	
c) If $D_{\vec{v}} f = 1$, then	$D_{-\vec{v}} f =$	

Solution:
a) ACEDFBG
b) 0,1,-1

Problem 4) (10 points)

The surface $f(x,y,z) = 1/10$ for $f(x,y,z) = 10z^2 - x^2 - y^2 + 100x^4 - 200x^6 + 100x^8 - 200x^2y^2 + 200x^4y^2 + 100y^4$ is a blueprint for a new sour-sweet gelatin candy brand.



a) (4 points) Find the equation $ax + by + cz = d$ for the tangent plane of f at $(0,0,1/10)$.

b) (3 points) Find the linearization $L(x,y,z)$ of f at $(0,0,1/10)$.

c) (3 points) Estimate $f(0.01, 0.001, 0.10001)$.

Solution:
a) The gradient is $[0,0,2]$ at the point. The equation is $2z = d$. The constant is obtained by plugging in the point. We get $2z = 2/10$ or $z = 1/10$.
b) The linearization is $1/10 + 2(z - 1/10)$.
c) We can estimate $1/10 + 2 \cdot 0.00001$.

Problem 5) (10 points)

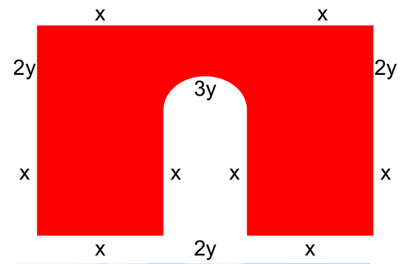
The **marble arch of Caracalla** is a Roman monument, built in the year 211. We look at a region modelling the arch. Using the Lagrange optimization method, find the parameters (x, y) for which the area

$$f(x, y) = 2x^2 + 4xy + 3y^2$$

is minimal, while the perimeter

$$g(x, y) = 8x + 9y = 33$$

is fixed.



Solution:
First write down the Lagrange equations $\nabla f = \lambda \nabla g$ and eliminate λ.

$$\begin{aligned} 4x + 4y &= \lambda 8 \\ 4x + 6y &= \lambda 9 \\ 8x + 9y &= 33 \; . \end{aligned}$$

We end up with $(4x + 4y)9 = 8(4x + 6y)\lambda$. Eliminating λ gives $y = x/3$. Plugging this into the constraint gives $\boxed{x = 3, y = 1}$.

Problem 6) (10 points)

a) (8 points) Find and classify the critical points of the function

$$f(x, y) = x^2 - y^2 - xy^3 \; .$$

b) (2 points) Decide whether f has a global maximum or minimum on the entire 2D plane.

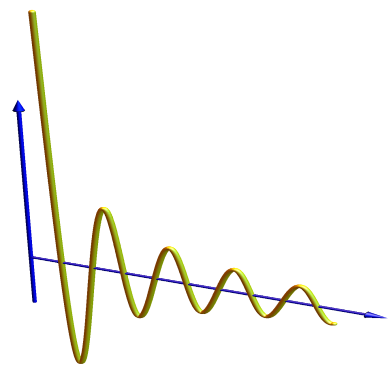
We don't know of any application for f. But if you read out the function aloud, it rolls beautifully off your tongue!

Solution:
a) The critical points are the solution of $2x - y^3 = 0, -2y - 3xy^2 = 0$. There is only one solution $(0,0)$. Computing $D = -4$ shows that the point is a saddle point.
b) Take $x = 1$, then we have $1 - y^2 - y^3$. this function has no maximum nor minimum. So, also in general this function has no global maximum nor global minimum on the entire plane.

Problem 7) (10 points)

We look at the integral

$$\int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} \sin(x)/x^2 \; dx dy \; .$$

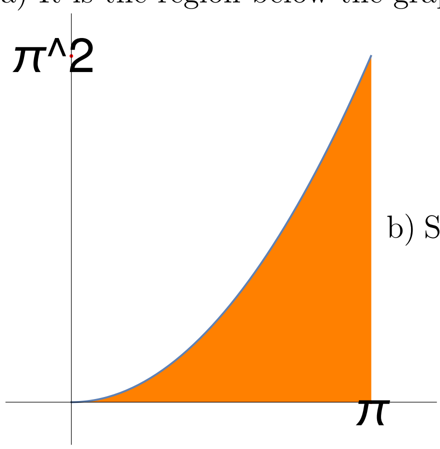


Just for illustration, we have drawn the graph of the function $f(x) = \sin(x)/x^2$.

a) (5 points) Draw the region over which the double integral is taken.

b) (5 points) Find the value of the integral.

Solution:
a) It is the region below the graph of $f(x) = x^2$ on the interval $[0, \pi]$.

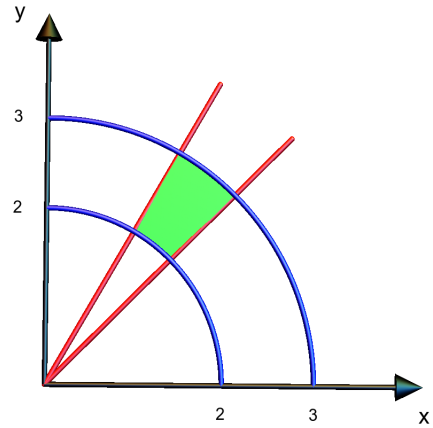


b) Switch the order of integration $\int_0^{\pi} \int_0^{x^2} \sin(x)/x^2 \; dydx = 2$.

Problem 8) (10 points)

"Heat-assisted magnetic recording" (HAMR) promises high density hard drives like 20 TB drives in 2019. The information is stored on sectors, now typically 4KB per sector. Let's assume that the magnetisation density on the drive is given by a function $f(x,y) = \sin(x^2+y^2)$. We are interested in the total magnetization on the sector R in the first quadrant bounded by the lines $x = y$, $y = \sqrt{3}x$ and the circles $x^2+y^2 = 4$ and $x^2+y^2 = 9$. In other words, find the integral

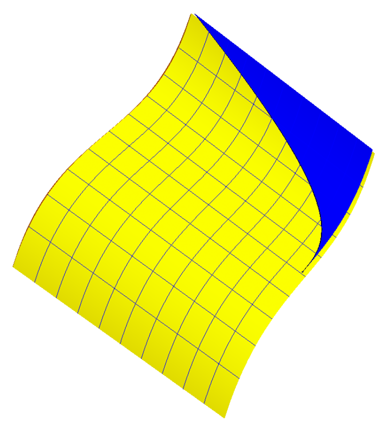
$$\int \int_R \sin(x^2+y^2)\; dxdy \; .$$



Solution:

$$\int_{\pi/4}^{\pi/3} \sin(r^2) r \; dr d\theta \; = (\pi/3 - \pi/4)[-\cos(r^2)/2|_2^3 = (\pi/24)(\cos(4) - \cos(9))$$

Problem 9) (10 points)



a) (4 points) Write down the double integral for the surface area of

$$\vec{r}(x,y) = [2x, y, x^3/3 \; + y]$$

with $0 \leq x \leq 2$ and $0 \leq y \leq x^3$.

b) (6 points) Find the surface area.

Solution:
a) $\int_0^2 \int_0^{x^3} \sqrt{x^4+8}\; dydx$.
b) $(24^{3/2} - 8^{3/2})/6$.