10/1/2019 FIRST HOURLY PRACTICE 5 **Math 21a, Fall 2019**

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) **T** F The vector $[0, 6/10, 8/10]$ is a direction = unit vector.

Solution:
Yes, its length is equal to 1.

2) T **F** Two nonzero vectors $\vec{v}$ and $\vec{w}$ are perpendicular if $\vec{v} \times \vec{w} = \vec{0}$.

Solution:
yes

3) T **F** For any vectors $\vec{u}$ and $\vec{v}$, we must have $\vec{v} \cdot \text{Proj}_{\vec{u}}\vec{v} = \vec{u} \cdot \text{Proj}_{\vec{v}}\vec{u}$.

Solution:
The left hand side depends on the length of v, the right hand side not.

4) **T** F The plane parametrized by $\vec{r}(t, s) = [t, s, 1]$ is the same as $z = 1$.

Solution:
Indeed

5) T **F** The surface $x^2 + y^2 - 2y - z^2 = 0$ is a cone.

Solution:
After completing the square there is a constant.

6) **T** F The volume of a parallelepiped generated by the vectors $\vec{u}, \vec{v}, \vec{w}$ is equal to $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$.

Solution:
Yes, this is a basic fact

7) | T | F | (T highlighted)

If a curve in space is parametrized by $\vec{r}(t)$ with $0 \leq t \leq 1$, then the same curve in the opposite direction can be parametrized by $\vec{r}(1-t)$ with $0 \leq t \leq 1$.

Solution:
The second parametrization reverses time but the curve is the same.

8) | T | F | (F highlighted)

The two-sheeted hyperboloid $x^2+y^2-z^2 = -1$ separates space into regions. The points $(3,4,6)$ and $(5,12,-14)$ lie in the same region.

Solution:
That was a tough cookie. One could think that because $g(x,y,z) = x^2+y^2-z^2$ has the same sign for both points that the regions are connected, but the regions are separated by a region in which the sign is different. Intuitively, one point is above the upper bowl, the second below the lower one.

9) | T | F | (T highlighted)

Given two vectors $\vec{u}$ and $\vec{v}$ which are perpendicular. Then $\text{Proj}_{\vec{u}}(\text{Proj}_{\vec{v}}\vec{w}) = \vec{0}$ for any vector $\vec{w}$.

Solution:

10) | T | F | (F highlighted)

The velocity vector $\vec{r}\,'(t)$ is always perpendicular to the curve.

Solution:
It is parallel to the curve

11) | T | F | (T highlighted)

If a point P with cylindrical coordinates (r,θ,z) and spherical coordinates (ρ,θ,ϕ) has the property that $r=\rho$, then it must be on the xy plane.

Solution:
Yes, $x^2+y^2 = x^2+y^2+z^2$ implies that $z=0$.

12) | T | F | (T highlighted)

The distance between the circle $x^2+y^2=1$ and $(x-3)^2+y^2=1$ is 1.

Solution:
Their centers have distance 3. Each circle has distance 1.

13) | T | F | The triple scalar product satisfies $\vec{u} \cdot (\vec{v} \times \vec{w}) \leq |\vec{u}||\vec{v}||\vec{w}|$.

Solution:
Use the formulas for the lengths.

14) | T | F | If the dot product between two vectors is positive, then the two vectors form an acute angle.

Solution:
Yes, if $\cos(\alpha) > 0$, then this means that $\alpha < \pi/2$.

15) | T | F | The surface given in cylindrical coordinates as $z^2 + r^2 = 1$ is a sphere.

Solution:
yes, it means $x^2 + y^2 + z^2 = 1$.

16) | T | F | The arc length of the curve $[\sin(t), \cos(t)]$ from $t = 0$ to $t = 1$ is equal to 1.

Solution:
Yes, the speed is equal to 1 so that the integral is 1.

17) | T | F | The curve $\vec{r}(t) = [\cos(t), \sin(t), t]$ hits the plane $z = 0$ at a right angle.

Solution:
The velocity vector is $-1, 0, 1$. This is not perpendicular to the plane.

18) | T | F | The lines $\vec{r}(t) = [t, -t, 2t]$ and $[5 - t, 3 + t, -2t$ are parallel.

Solution:
Yes they have both the same velocity vector.

19) | T | F | The parametrized curve $[0, 7\cos(1+t), 3\sin(1+t)]$ is an ellipse.

Solution:
Indeed, it is part of the yz-plane.

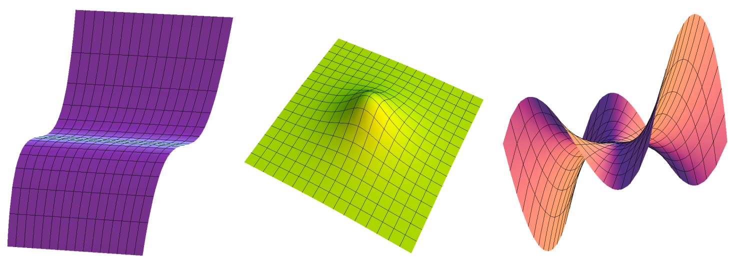
20) | T | F | $\vec{u} \times (\vec{v} \times \vec{u}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

Solution:
Take $u = i, v = j$.

Total

Problem 2) (10 points) No justifications are needed here.

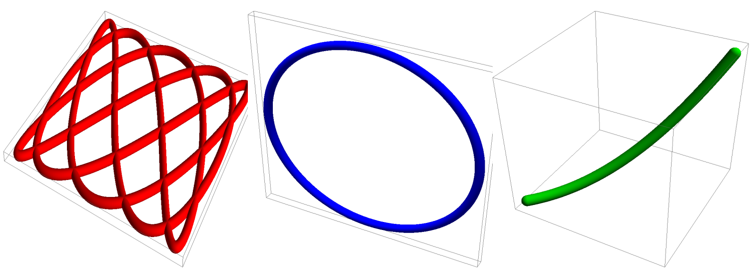
a) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.



I II III

Function $f(x,y) =$	Enter O,I,II or III
$x^3 - xy^2$	
y^3	
$1/(1+x^2+y^2)$	
$x^4 + y^4$	

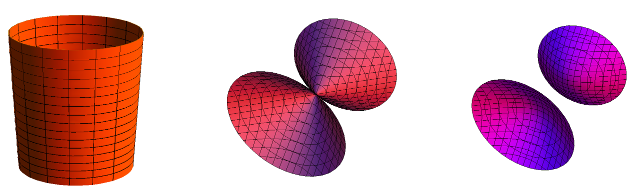
b) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



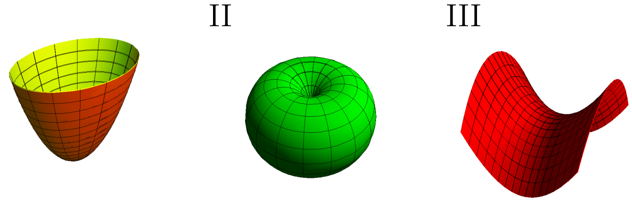
I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\vec{r}(t) = [\cos(3t), \sin(5t), 0]$	
$\vec{r}(t) = [t, t, t^2]$	
$\vec{r}(t) = [\cos(t), 0, \sin(t)]$	
$\vec{r}(t) = [\sin(t), \sin(t), \sin(t)]$	

c) (4 points) Match the surfaces to the pictures. There is an exact match here.



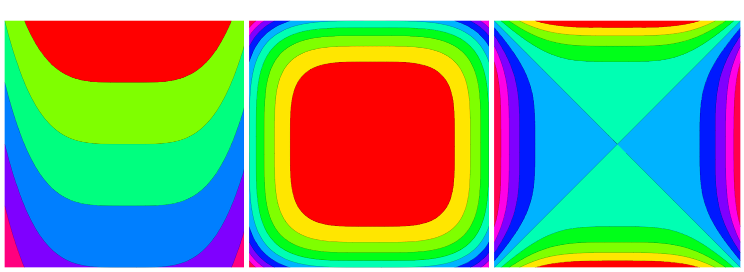
I II III



IV V VI

Description	I,II,III,IV,V,VI
$[2u\cos(v), 4u\sin(v), u^2]$	
$[u^3, v^3, u^6 - v^6]$	
$\rho = \sin(\phi)$	
$r = 1$	
$x^2 - y^2 + z^2 = -1$	
$x^2 = y^2 - z^2$	

d) (2 points) Match the contour maps for $f(x,y)$. Enter O if no match.



I II III

function $f(x,y) =$	O,I,II,III
$f(x,y) = x^4 + y^4$	
$f(x,y) = x^4 - y^4$	
$f(x,y) = x - y$	
$f(x,y) = x^4 - y$	

Solution:
III,I,II,0
I,III,II,0
IV,VI,V,I,III,II
II,III,O,I

Problem 3) (10 points)

The front roof line of the "spider" on the Harvard lecture halls forms a line

$$\vec{r}(t) = [1+t, 2+t, 1] \ .$$

On top of the telescope sits a fly at the point $P = (0,0,10)$. Find the distance of P to the line.

Solution:
The line contains the point $Q = (1,2,1)$ and the vector $\vec{v} = [1,1,0]$. We need $\vec{PQ} = [1,2,-9]$ and $\vec{v}$. We have $\vec{v} \times \vec{PQ} = [-9,9,1]$ which has length $\sqrt{163}$. The distance is $\boxed{\sqrt{163}/\sqrt{2}}$.

Problem 4) (10 points)

The kinect sensor can be used to scan objects. An infrared laser is used to measure distances in the horizontal plane.

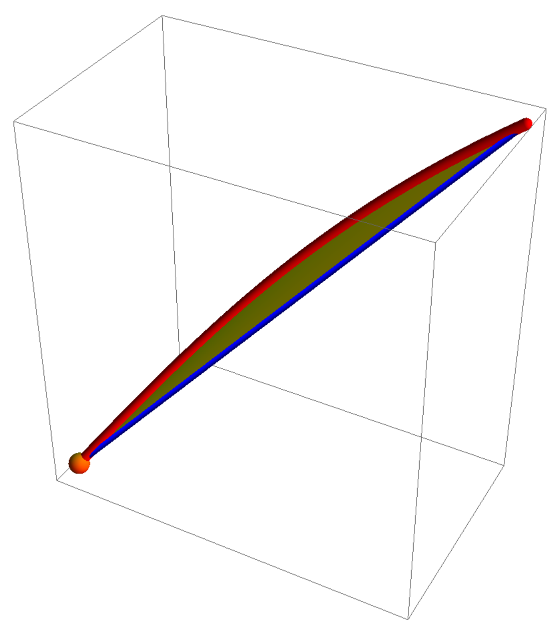
a) (2 points) Find an equation which tells that a point $P = (x, y)$ has distance 5 from the sensor $(0, -1)$.

b) (2 points) Find an equation which tells that a point $P = (x, y)$ has distance 4 from the sensor $(0, 1)$.

c) (6 points) Assume we know that P has distance 5 from $(0, -1)$ and distance 4 from $(0, 1)$. Where is this point (x, y) if we assume that it has a positive x-coordinate?

Solution:
a) $x^2 + (y + 1)^2 = 25$.
b) $x^2 + (y - 1)^2 = 16$.
c) Take both equations and subtract to get$(y+1)^2 - (y-1)^2 = 9$.This gies $y = 9/4$. From one of the equations we get $x = \sqrt{231}/4$.

Problem 5) (10 points)



a) (6 points) Given $\vec{r}(t) = [t + t^3/3, \arctan(t), \sqrt{2}t]$. Find the arc length from $t = 0$ to $t = 1$.

b) (4 points) Compute the vector integral

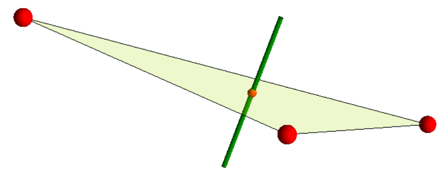
$$\int_0^1 \vec{r}\,'(t)\;dt$$

by integrating coordinate by coordinate. Verify that the length of this vector agrees with the arc length of the straight line connecting $\vec{r}(0)$ with $\vec{r}(1)$.

Solution:
a) $\vec{r}\,'(t) = [1+t^2, 1/(1+t^2), \sqrt{2}]$ so that $|\vec{r}\,'(t)| = \sqrt{(1+t^2)^2 + 1/(1+t^2)^2 + 2}$ which simplifies to $|\vec{r}\,'(t)| = 1 + t^2 + (1+t^2)^{-1}$. Integrating this from 0 to 1 gives $\boxed{4/3 + \pi/4}$.
b) If we integrate $\vec{r}\,'(t) = [1+t^2, 1/(1+t^2), \sqrt{2}] = \vec{r}(t)$, we get $[t + t^3/3, \arctan(t), \sqrt{2}t]$ and evaluating this at $t = 0$ and $t = 1$. The length of this vector is just the distance of the two points.
Remark. This problem had the purpose to make you aware what the difference is between the arc length $\int_a^b |\vec{r}\,'(t)|\, dt$ and $|\int_a^b \vec{r}\,'(t)|$, which is the distance between the two end points. Of course the later is always smaller or equal than the former.

Problem 6) (10 points)

Given four points $A = (1,2,1), B = (1,0,1), C = (0,1,1), D = (1,1,2)$.



a) (4 points) Find an equation $ax + by + cz = d$ for the plane which contains A, B, C.
b) (3 points) Parametrize the line L which passes through D perpendicular to the plane ABC.
c) (3 points) Where does L hit the plane through A, B, C?

Solution:
a) We take the cross product of $\vec{AB} = [0,-2,0]$, $\vec{AC} = [-1,-1,0]$ to get $\vec{AB} \times \vec{AC} = [0,0,-2]$. The equation of the plane is $-2z = d$ where d is a constant. We can get d by pluggin in a point. We have $-2z = -2$ or $\boxed{z = 1}$.
b) $\vec{r}(t) = [1,1,2] + t[0,0,-2] = [1,1,2-2t]$.
c) In order to get the intersection with the plane, we have to see where $2 - 2t = 1$, this gives $t = 1/2$. The point $\boxed{\vec{r}(1) = [1,1,1]}$ is the intersection point.

Problem 7) (10 points)

British stuntman **Gary Connery** made aviation history last year by becoming the first skydiver to land without parachute. He landed in 18000 boxes. Assume he started with an initial velocity $[0, 100, 0]$ from the initial point $[0, 0, 800]$. He was exposed to an acceleration $\vec{r}\,''(t) = [0, 0, -10 + t]$. Where is his location at time t=6?

Solution:
Start with the acceleration, then integrate to get the velocity, then integrate again to get the position. In each step we add a constant so that the setup is correct for $t = 0$, where the initial velocity and position are given.

$$\vec{r}\,''(t) = [0, 0, -10 + t] \ .$$

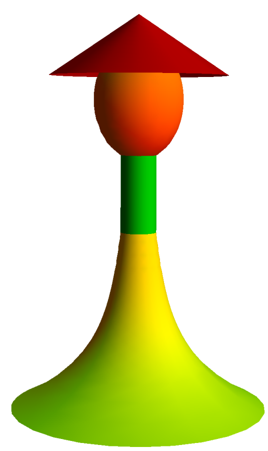
$$\vec{r}\,'(t) = [0, 100, -10t + t^2/2] \ .$$

$$\vec{r}\,(t) = [0, 100t, 800 - 5t^2 + t^3/6] \ .$$

Now plug in $t = 6$ to get $\boxed{[0, 600, 656]}$.

Problem 8) (10 points)

We parametrize the queen in a fancy chess set. It consists of 5 surfaces. Parametrize them. You do not have to give bounds for the parameters. In each case, just give an answer of the form $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ without further explanations.



a) (2 points) "hat" Cone $x^2 + y^2 = (1 - z)^2$.

b) (2 points) "head" Sphere $x^2 + y^2 + (z + 1/2)^2 = 1$.

c) (2 points) "neck" Cylinder $x^2 + y^2 = 1/4$.

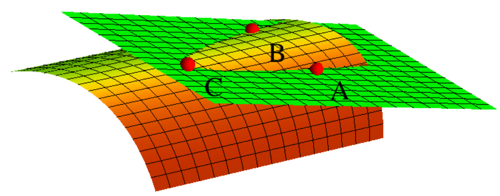
d) (2 points) "robe" Hyperboloid $x^2 + y^2 - (z + 4)^2 = 1$.

e) (2 points) "floor" Plane $z = -8$

Solution:
a) Since the relation between z and r is $r = 1 - z$, we have $[(1-v)\cos(u), (1-v)\sin(u), v]$.
b) We just have to shift the last coordinate a bit from the standard parametrization of the sphere: $[\sin(v)\cos(u), \sin(v)\sin(u), \cos(v) - 1/2]$.
c) The cylinder has radius 1/2 so that $[\cos(u)/2, \sin(u)/2, v]$.
d) This is a surface of revolution. We just have to see that the equation tells us that $r^2 = 1 + (z+4)^2$, so that $[\sqrt{1+(v+4)^2}\cos(u), \sqrt{1+(v+4)^2}\sin(u), v]$.
e) This is best done as a graph: $[u, v, -8]$.

Problem 9) (10 points)

We are given a surface parametrized as $\vec{r}(u,v) = [u+v, u^2, v]$.



a) (2 points) Locate the points $A = \vec{r}(1,2), B = \vec{r}(-1,2)$ and $C = \vec{r}(0,0)$.

b) (4 points) Parametrize the plane through A, B, C.

c) (4 points) Find the area of the triangle with vertices A, B, C.

Solution:
a) $A = (3,1,2), B = (1,1,2), C = (0,0,0)$.
b) $\vec{r}(u,v) = [3,1,2] + t[2,0,0] + s[3,1,2]$.
c) $[2,0,0] \times [3,1,2] = [0,-4,2]$ which has length $2\sqrt{5}$. The answer is $\boxed{\sqrt{5}}$.