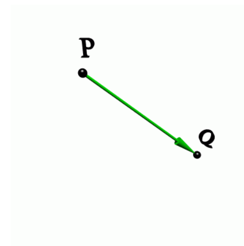
Distances overview

DISTANCE POINT-POINT (3D). If P and Q are two points, then

$$d(P,Q) = |\vec{PQ}|$$

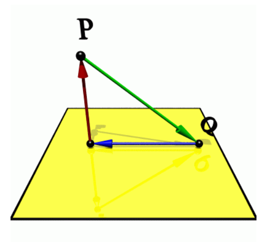
is the distance between P and Q. We use the notation $|\vec{v}|$ instead of $||\vec{v}||$ in this handout.



DISTANCE POINT-PLANE (3D). If P is a point in space and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P,\Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

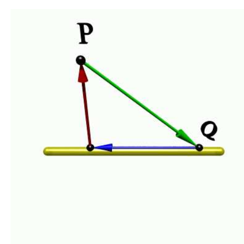
is the distance between P and the plane. Proof: you can see this as a scalar projection of $\vec{PQ}$ onto $\vec{n}$. (P.S. If the plane is parametrized $\vec{r} = \vec{OQ} + t\vec{v} + s\vec{w}$, find first $\vec{n} = \vec{v} \times \vec{w}$.)



DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $\vec{r}(t) = 0\vec{Q} + t\vec{u}$, then

$$d(P,L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

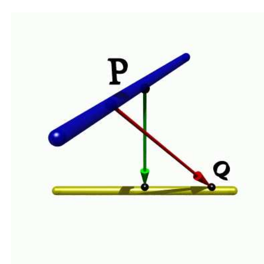
is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram.



DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

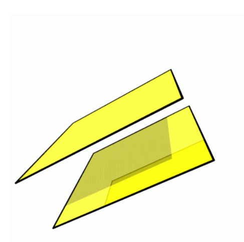
is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines.



DISTANCE PLANE-PLANE (3D). If $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$ are two parallel planes, then their distance is

$$\frac{|e-d|}{|\vec{n}|} .$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane.

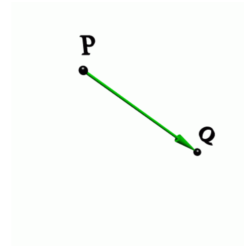


EXAMPLES

DISTANCE POINT-POINT (3D). $P = (-5, 2, 4)$ and $Q = (-2, 2, 0)$ are two points, then

$$d(P, Q) = |\vec{PQ}| = \sqrt{(-5+2)^2 + (2-2)^2 + (0-4)^2} = 5 .$$

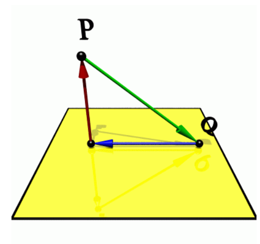
A question: what is the distance between the point $(-5, 2, 4)$ and the sphere $(x+2)^2 + (y-2)^2 + z^2 = 1$?



DISTANCE POINT-PLANE (3D). $P = (7, 1, 4)$ is a point and $\Sigma : 2x+4y+5z = 9$ is a plane which contains the point $Q = (0, 1, 1)$. Then

$$d(P, \Sigma) = \frac{|\langle -7, 0, -3\rangle \cdot \langle 2, 4, 5\rangle|}{|\langle 2, 4, 5\rangle|} = \frac{29}{\sqrt{45}}$$

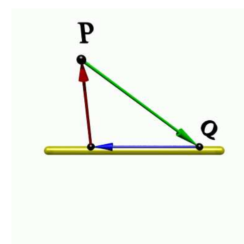
is the distance between P and Σ. A Question: without the absolute value, the result would have been negative. What does this tell about the point P?



DISTANCE POINT-LINE (3D). $P = (2, 3, 1)$ is a point in space and L is the line $\vec{r}(t) = (1, 1, 2) + t(5, 0, 1)$. Then

$$d(P, L) = \frac{|\langle -1, -2, 1\rangle \times \langle 5, 0, 1\rangle|}{\langle 5, 0, 1\rangle} = \frac{|\langle -2, 6, 10\rangle|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}$$

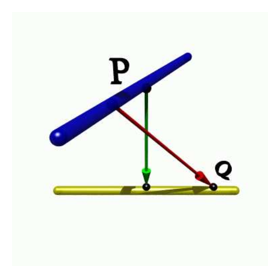
is the distance between P and L. Question to the reader: what is the equation of the plane which contains the point P and the line L?



DISTANCE LINE-LINE (3D). L is the line $\vec{r}(t) = (2, 1, 4) + t(-1, 1, 0)$ and M is the line $\vec{s}(t) = (-1, 0, 2) + t(5, 1, 2)$. The cross product of $\langle -1, 1, 0\rangle$ and $\langle 5, 1, 2\rangle$ is $\langle 2, 2, -6\rangle$. The distance between these two lines is

$$d(L, M) = \frac{|(3, 1, 2) \cdot (2, 2, -6)|}{|\langle 2, 2, -6\rangle|} = \frac{4}{\sqrt{44}} .$$

Question to the reader: also here, without the absolute value, the formula can give a negative result. What happens with this sign, when P and Q are interchanged?



DISTANCE PLANE-PLANE (3D). $5x + 4y + 3z = 8$ and $5x + 4y + 3z = 1$ are two parallel planes. Their distance is

$$\frac{|8-1|}{|\langle 5, 4, 3\rangle|} = \frac{7}{\sqrt{50}} .$$

Question for the reader: what is the distance between the planes $x + 3y - 2z = 2$ and $5x + 15y - 10z = 30$?