Homework 9: Functions and Continuity

This homework is due Friday, 9/29 rsp Tuesday 10/3.

1 Determine which f extends to a continuous function in the entire plane. No reasoning is required. Each question one point. log = ln denotes the natural log.

a) $f(x,y) = xy(x^2-4)/(x+2)$.
b) $f(x,y) = \log(1+|xy|)$
c) $f(x,y) = (x^2-y^2)/x^2+y^2)$
d) $f(x,y) = \sin(x^2+y^2)$
e) $f(x,y) = 1/\log(2+|x+y|)$
f) $f(x,y) = y\exp(1/x)$
g) $f(x,y) = \log(\exp(x+y))$
h) $f(x,y) = \sin(ye^{\cos(x)})$
i) $f(x,y) = \sin(y)\exp(x)$
j) $f(x,y) = \exp(\log|x+y|)$

2 Investigate whether the following functions are continuous at $(0,0)$. This means: investigate whether a function value at $(0,0)$ exists which extends the definition, so that the extended function is continuous. If the limit $(x,y) \to (0,0)$ exists, tell what the limit is. a) $f(x,y) = x^2y/(x^2+y^2)^2$. b) $f(x,y) = xy/(x^2+y^2)$. It can be

helpful to use polar coordinates. You have to give some reasoning here.

3 Find the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists or show that the limit does not exist

a) $f(x,y) = \frac{6x^4y}{2x^5+y^5}$ b) $f(x,y) = \frac{x^6-y^6}{(x^2+y^2)^2}$

Also here is some reasoning required.

4 Determine the set of points where the following function is continuous $f(x,y) = \frac{e^x+e^y}{e^{x+y}-1}$.

5 Find the limit $(x,y) \to (0,0)$ of the function $f(x,y) = \sin(x^2+y^2)\log(x^2+y^2)$, where $\log(x) = \ln(x)$.

Main definition

A function $f(x,y)$ with domain R is **continuous at** $(a,b) \in R$ if $f(x,y) \to f(a,b)$ for all choices $(x,y) \to (a,b)$. We also say that f **continuous at a point** (a,b) **not in the domain** if there exists a finite value $f(a,b)$ such that $f(x,y) \to f(a,b)$ whenever $(x,y) \to (a,b)$ for $(x,y) \in R$. For example, the function $f(x,y) = y(x^2-1)/(x-1)$ continuous everywhere even so $x = 1$ is not in the domain. We can fill in the value $f(1,y) = 2y$ as the function is equivalent to $y(x+1)$, its analytic continuation. Also $f(x,y) = \sin(x^2+y^2)/(x^2+y^2)$ is continuous as l'Hopital for polar coordinates shows: with filling in the whole $f(0,0) = 1$, it becomes continuous even at the point $(0,0)$. In one dimensions, there are **jump discontinuities** like $f(x) = \text{sign}(x)$ or **poles** like $f(x) = 1/x$ or **oscillations** like $f(x) = \sin(1/x)$. These three prototypes can happen in the same function like in $1/\sin(1/x))$ or $\arctan(1/\sin(1/x))$. Many questions about continuity in two dimensions are answered when writing the function in polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ near the point in question. For $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ for example, the function becomes (just fill in $x = r\cos(\theta), y = r\sin(\theta)$), in polar coordinates $f(r,\theta) = \cos(2\theta)$. The value of the function depends only on the angle. Arbitrarily close to $(0,0)$, the function takes any values between -1 and 1. The function is not continuous because no value can be found at $(0,0)$ such that f can be continuously extended to it.