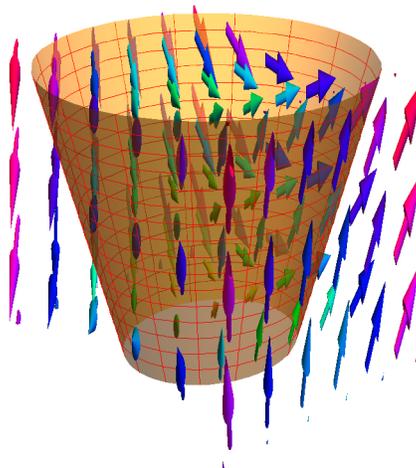


## Homework 30: Stokes Theorem

This homework is due Wednesday, 11/29 or Tuesday 11/28 after thanksgiving.

- 1 Evaluate  $\int_S \text{curl}(\mathbf{F}) \cdot d\vec{S}$ , where  $\vec{F}(x, y, z) = \langle -3y, x, z + \sin(z - 1) \sin(z - 2) \rangle$ , and where  $S$  is the part of the cone  $z^2 = x^2 + y^2$  that lies between the planes  $z = 1$  and  $z = 2$ . The cone can be parametrized by  $\vec{r}(u, v) = \langle v \cos(u), v \sin(u), v \rangle$  with  $0 \leq u \leq 2\pi$  and  $1 \leq v \leq 2$ .



- 2 Evaluate the line integral  $\int_C \vec{F} \cdot d\mathbf{r}$ , where  $\vec{F}(x, y, z) = \langle x^3 - 4e^{-\cos(x)}, 3e^x, z^5 + e^z \rangle$  and where  $C$  is the boundary of the part of the plane  $2x + y + z = 5$  in the first octant, oriented counterclockwise as viewed from above.
- 3 Evaluate the line integral  $\int_C \vec{F} \cdot d\mathbf{r}$ , where  $\vec{F}(x, y, z) = \langle xy, 2z, 3y \rangle$  and  $C$  is the curve of intersection of the plane  $x + z = 5$  and the cylinder  $x^2 + y^2 = 9$ , oriented counterclockwise as viewed from above.
- 4 Compute both sides of Stokes' Theorem for  $\vec{F}(x, y, z) = \langle -2yz, y, 3x \rangle$  and the surface  $S$  which is the part of the paraboloid  $z = 5 - x^2 - y^2$  that lies above the plane  $z = 1$ , oriented upwards.

- 5 a) Evaluate  $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$  with  $\vec{F}(x, y, z) = \langle y + \sin x, z^2 + \cos y, x^3 \rangle$ , where  $C$  is the curve  $\vec{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$ ,  $0 \leq t \leq 2\pi$  which as you can see lies on the surface  $z = 2xy$ .
- b) Explain without doing any computation that if  $S$  is the torus  $\vec{r}(u, v) = \langle (2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v) \rangle$  with  $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$  and  $\vec{F}$  is a vector field like  $\vec{F}(x, y, z) = \langle e^{e^{e^x}}, \sin \sin(y + z + x), x^{100} \rangle$  then  $\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$ .

## Main points

**Stokes theorem:** Let  $S$  be a surface bounded by a curve  $C$  and  $\vec{F}$  be a vector field. Then

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$

The orientation of  $S$  is given by the parametrization: the orientation of  $C$  is such that if you walk along  $C$  with the head in the "up" direction  $\vec{r}_u \times \vec{r}_v$  and your nose into the  $\vec{r}'$  direction, then your left foot is on the surface.

Written out in detail, we have

$$\int \int_R \text{curl}(\vec{F}(\vec{r}(u, v))) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

A lot of things come together here: surfaces, curves, dot product, cross product, triple scalar product, vector fields, double integrals and curl. What does it mean? From a SMBC cartoon: **"Stokes theorem? Yeah, thats how if you draw a loop around something, you can tell how much swirly is in it."**