

Homework 29: Flux integral, Stokes I

This homework is due Monday, 11/20 resp Tuesday 11/21 just before Thanksgiving. If no orientation is given for a surface, the orientation is assumed to be "outwards". There is one problem which gives a first exposure to Stokes theorem The Monday/Tuesday lecture before thanksgiving will cover the theorem again.

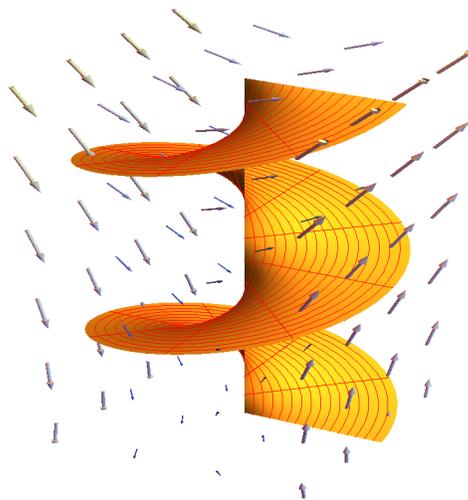
- 1 Evaluate the flux integral $\int \int_S \vec{F} \cdot d\vec{S}$ if

$$\vec{F}(x, y, z) = \langle 7x, y^2, z \rangle ,$$

and S is the helicoid

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, 0 \leq u \leq 1, 0 \leq v \leq 6\pi$$

which has an upward orientation.



- 2 Evaluate the flux integral $\int \int_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle x, y, 5 \rangle ,$$

where S is part of the cylinder $x^2 + z^2 = 1$ bound by $y = 0$ and $y = 7$. The surface is oriented outwards.

- 3 The temperature $f(x, y, z)$ at a point (x, y, z) is equal to the distance from the origin $(0, 0, 0)$. Find the flux of the heat gradient field $\vec{F} = -\nabla f$ across a sphere S of radius 2 centered at $(0, 0, 0)$.

4 Let $\vec{F}(x, y, z)$ be an inverse square field, that is

$$\vec{F}(x, y, z) = c\langle x, y, z \rangle / \rho^3$$

with $\rho = \sqrt{x^2 + y^2 + z^2}$. Show that the flux of \vec{F} across a sphere S with center at the origin and radius R is independent of the radius of the sphere.

5 Use Stokes theorem to evaluate the flux integral $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle (x^9 + y^7)z^5, x, y \rangle ,$$

where S is the surface $x^2 + y^2/16 + z^8 = 1, z \geq 0$, oriented upwards. Stokes theorem expresses this as a line integral along the boundary curve $\vec{r}(t) = \langle \cos(t), 4 \sin(t), 0 \rangle, 0 \leq t \leq 2\pi$.

Main points

If a surface S is parametrized as $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over a domain G in the uv -plane and \vec{F} is a vector field, then the **flux integral** of \vec{F} through S is

$$\int \int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv .$$

With $d\vec{S} = (\vec{r}_u \times \vec{r}_v) \, dudv$, this can be written as $\int \int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if $\vec{F} =$ fluid velocity field, then $\int \int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through S in unit time.

Stokes theorem tells that if S be a surface bounded by a curve C and \vec{F} be a vector field, then

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$