Homework 28: Curl and Div

This homework is due Friday, 11/17 rsp Tuesday 11/21.

1 a) Find the curl of the vector field

$$\vec{F}(x,y,z) = \langle \sin(y+z), \sin(z+x), \sin(x+y) \rangle \ .$$

b) Find the divergence of the gradient of $f(x,y,z) = x^3 + y^5 + 3z^2y$ c) Define $\Delta f = f_{xx} + f_{yy} + f_{zz}$. After having done a) and b), evaluate $\text{div}(\text{curl}(\vec{F}))$, Δf at $(1,1,1)$ and $\text{div}(\text{grad}(f)$ and $\text{curl}(\text{grad})(f)$ **in your head**! There is no need here to write down anything than the three numbers.

2 Let f be a scalar field and $\vec{F}$ a vector field in space. Determine which expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. In all problems, we deal with functions and vector fields in 3D space.

a)	$\text{curl}(\text{div}(\vec{F}))$	b)	$\text{curl}(\text{grad}(f))$
c)	$\text{grad}(\vec{F})$	d)	$\text{grad}(\text{div}(\vec{F})$
e)	$\text{div}(\text{grad}(f))$	f)	$\text{grad}(\text{div}(f))$
g)	$\text{curl}(\text{curl}(\vec{F}))$	h)	$\text{div}(\text{div}(\vec{F}))$
i)	$\text{grad}(f) \times \text{div}(\vec{F})$	j)	$\text{div}(\text{curl}(\text{grad} f))$
k)	$\text{curl}(f)$	l)	$\text{curl}(\text{div}(f))$

3 a) Is there a vector field $\vec{G}(x,y,z)$ such that $\text{curl}(\vec{G}) = \langle 8, 7, 12 \rangle$. If yes, find one.
b) Is there a vector field $\vec{G}(x,y,z)$ such that

$$\text{curl}(\vec{G}) = \langle xyz, -y^2z, yz^2 \rangle \ ?$$

If yes, find one.
c) Assume $\vec{F}$ is a gradient field. Does this imply that there is a vector field $\vec{G}$ such that $\text{curl}(\vec{G}) = \vec{F}$? If yes, show it. If no, find a counter example.

4 a) Verify that any vector field of the form

$$\vec{F}(x,y,z) = \langle f(x), g(y), h(z) \rangle$$

is irrotational.

b) Verify that any vector field of the form

$$\vec{F}(x,y,z) = \langle f(y,z), g(x,z), h(x,y) \rangle$$

is incompressible.

c) Find a nonzero vector field $\vec{F}$ such that $\text{curl}(\vec{F}) = \langle 0,0,0 \rangle$.

5 a) Prove the identity $\text{div}(\nabla f \times \nabla g) = 0$.

b) Show that the scalar function $f(x,y,z) = 3\sin(z) + z^5 + y + x$ is the divergence of some vector field $\vec{F}$.

Main points

The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is the vector field $\text{curl}(P,Q,R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$.

The curl measure rotation of a field. If $\vec{F}$ has zero curl everywhere it is **irrotational**. Remember that in two dimensions, the curl of $\vec{F} = \langle P, Q \rangle$ is a scalar.

The **divergence** of a vector field $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ is

$$\text{div}(\vec{F})(x,y,z) = P_x(x,y,z) + Q_y(x,y,z) + R_z(x,y,z).$$

$\text{div}(\vec{F})$ measures the expansion of a field. Zero divergence everywhere is called **incompressible**.

$\text{div}(\text{curl}((\vec{F})) = 0$ for all vector fields $\vec{F}$. $\text{curl}(\nabla f) = 0$ for all functions f.