Homework 27: Green's theorem

This homework is due Wednesday, 11/15 rsp Thursday 11/16.

1 Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \ ,$$

where $\vec{F}(x,y) = \langle 123y + 3x^{15} - 1, 17x + \cos(9y)y^{300}\rangle$ and C consists of the line segments from $(0,2)$ to $(0,0)$ and from (0,0) to (2,0) and the curve $y = \sqrt{4-x^2}$ from $(2,0)$ to $(0,2)$.

2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ with

$\vec{F}(x,y) = \langle y^2\cos(x)+\sin(\sin(x)), 45x+x^3+2y\sin(x)+\sin(\sin(y))\rangle$

where C is the triangular path from (0,0) to (2,6) to (2,0) to (0,0). Watch the orientation of the curve!

b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x,y) = \langle -8y + 1/(1+x+y^3), 7x + 3y^2/(1+x+y^3)\rangle$$

and C is the circle $(x-6)^2 + (y-7)^2 = 49$ oriented counterclockwise.

3 A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 4\cos^3(t), 4\sin^3(t)\rangle, 0 \leq t \leq 2\pi \ .$$

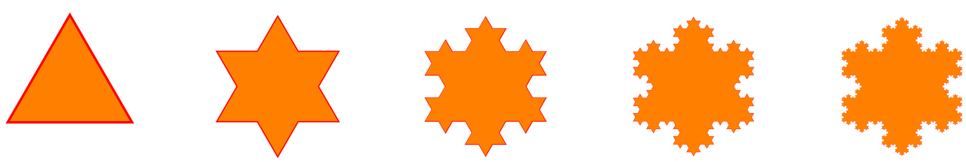
We can not do that directly. Guess which theorem to use, then use it!

4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \langle x^2 + y, 3x - \sin(y^2)\rangle$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise.

5 Use Green's Theorem to evaluate

$$\int_C \langle \sin(\sqrt{1+x^3}), 7x \rangle \, d\vec{r} \,,$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length 1/3 added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length 1/9 added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length 1/27 added and $K(4)$ is $K(3)$ with $3*4^3$ triangles of length 1/81 added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26\ldots$ which is between 1 and 2.



Main points

The **curl** of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \,.$$

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field and G is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \iint_G \text{curl}(F) \, dxdy \,.$$