# Homework 26: Theorem of line integrals

This homework is due Monday, 11/13 rsp Tuesday 11/14.

**1** a) Find a function $f$ such that $\vec{F} = \nabla f$ if $\vec{F}(x, y) = \langle x^8 - 2xy^2 + y, y^5 - 2x^2y + x \rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.

**2** a) Find a function $f$ such that $\vec{F} = \nabla f$ if $\vec{F}(x, y) = \langle 30y^2/(1 + x^2), 60y \arctan(x) \rangle$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = \langle t^2, 2t \rangle$ with $0 \leq t \leq 1$.

**3** On August 1, 2017, Lukas Irmler walked over a rope over the **Rheinfalls** in Switzerland.
There is a force field $\vec{F}$ present which consists part of the gravitational force and part by the wind forces: $\vec{F}(x, y, z) = \langle \sin(x), \cos(y), -10 + z \rangle$. The path is given by $\vec{r}(t) = \langle 5t, t, 30 - \sin(t)/10 \rangle$, where $0 \leq t \leq \pi$. Compute the work $\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ done by Lukas during this stunt.

**4** a) Verify that if $\vec{F} = \langle P, Q, R \rangle$ is conservative, then

$$P_y = Q_x, P_z = R_x, Q_z = R_y \, .$$

b) Is $\langle x^5y, xy^2, zx \rangle$ conservative? If yes, find $f$ such that $\vec{F} = \nabla f$, if not, give a reason.

**5** a) Show that the vector field $\vec{F}(x, y, z) = \langle y, x, xyz \rangle$ is not conservative by using problem 4).
b) Find two different curves $C_1, C_2$ from $(0, 0, 0)$ to $(1, 1, 0)$ for which the line integrals of $\vec{F}$ along $C_1, C_2$ are different.

## Main points

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions. It tells that the work done along a path is the potential energy difference.

**Fundamental theorem of line integrals**: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

This theorem can be used to dramatically simplify the computation of a line integral. Just find the potential $f$ and evaluate the difference of potential values.

Recall that a region $R$ is called **simply connected** if every closed loop in $R$ can be pulled together to a point within $R$.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:
Gradient field ↔ Conservative ↔ Closed loop property
In simply connected open regions, these three properties are all equivalent to being irrotational $\mathrm{curl}(\vec{\mathrm{F}}) = \mathrm{Q_x} - \mathrm{P_y} = 0$.