Homework 25: Line integrals

This homework is due Friday, 11/10 rsp Tuesday 11/14.

1 a) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x, y, z) = \langle x^3 + y^3, y - z, z^2 + 1 \rangle$ and $\vec{r}(t) = \langle 11t^2, 11t^3, 11t \rangle$ with $0 \leq t \leq 3$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x, y, z) = \langle 15z, 15y, -15x \rangle$ and $\vec{r}(t) = \langle 2t, \sin(t), \cos(t) \rangle,\ 0 \leq t \leq \pi$.

2 An electric current I produces a magnetic field $\vec{B}$ whose flow lines are circles circling the wire. Let $C : \langle r\cos(t), r\sin(t), 0 \rangle$. Ampères law is $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$, where μ_0 is a constant called permeability. Show that the magnitude $B(r) = |\vec{B}|$ of the magnetic field at a distance r from the center of the wire is $B = \frac{\mu_0 I}{2\pi r}$. Note that B is a scalar function and $\vec{B}$ is a vector field. Use first the information provided to find the vector field $\vec{B}$.

3 Determine from each of the following cases, whether $\vec{F}$ is conservative (a gradient field or not). If it is, find a function f such that $\vec{F} = \nabla f$.

a) $\vec{F}(x, y) = \langle y + 4x^3 + y^6, -x + 6x^4 y^5 \rangle$
b) $\vec{F}(x, y) = \langle x + 7e^x \sin(y), y^4 + 7e^x \cos(y) \rangle$
c) $\vec{F}(x, y, z) = \langle x + y, y + x, z^5 - \sin(z) \rangle$
d) $\vec{F}(x, y, z) = \langle x^5, z^5, y \rangle$

4 Evaluate the line integral $\int_C \langle 1 - ye^{-x}, e^{-x} \rangle \cdot d\vec{r}$, where C is the path $r(\vec{t}) = \langle t, 1 + t + \sin(\sin(t)) \rangle$ and t is from 0 to π. You probably will have difficulty. A future "you" (who has a time machine) tells you that you can compute the integral also in a different way: find a function f which is a potential to the vector field, then evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0))$. You can use this without justification for now. We will learn about this "warp" feature in the next lecture.

5 The topological notions appearing in this problem are cool but not very essential for the course. They play a role next week. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
a) $\{(x,y) \,|\, 1 < y < 3\}$, b) $\{(x,y) \,|\, 1 < |x| < 2\}$
c) $\{(x,y) \,|\, 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$ d) $\{(x,y) \,|\, (x,y) \neq (1,2)\}$
e) $\{(x,y,z) \,|\, (x,y,z) \neq (1,2,3)\}$
f) $\{(x,y,z) \,|\, (x,y,z) \neq \{(\cos(t), \sin(t), 0) \mid 0 \leq t \leq 2\pi\}$

Main definitions

If $\vec{F}$ is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a,b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is called the **line integral** of $\vec{F}$ along the curve C. The field $\vec{F}$ is **conservative** in a region R if the line integral from A to B is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. It is **irrotational** if $\text{curl}(F) = Q_x - P_y$ is zero everywhere in R.

A subset G of the plane is **open** if every point (x,y) in G is contained in a small disc D centered at (x,y) and $D \subset G$. (Openness is useful to make sure we do not hit a boundary where one has to worry about differentiability). G is **connected**, if one can connect any two points in G with a curve within G. it is **simply connected** if it is connected and every closed curve in G can be deformed to a point within G.

Clairaut test: Zero curl is necessary for a gradient field.