

Homework 23: Cylindrical/Spherical integration

This homework is due Monday, 11/6 rsp Tuesday 11/7.

1 Evaluate the following integrals.

a) $\int_0^\pi \int_0^2 \int_0^{9-r^2} r \, dz \, dr \, d\theta$

b) $\int_0^{2\pi} \int_{\pi/2}^\pi \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

2 Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

3 Use spherical coordinates to evaluate

$$\iiint_H (16 - x^2 - y^2) \, dV ,$$

where H is the solid hemisphere $x^2 + y^2 + z^2 \leq 16$, $z \geq 0$.

4 Evaluate the integral by changing to spherical coordinates.

$$\int_{-a}^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2 z + y^2 z + z^3) \, dz \, dx \, dy$$

5 Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)} \, dx \, dy \, dz = 2\pi .$$

Main definitions

The integration factor in cylindrical coordinates $(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$ is r as in polar coordinates.

The integration factor in spherical coordinates $(x, y, z) = (\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))$ is $\rho^2 \sin(\phi)$. This was the surface area element $|\vec{r}_\phi \times \vec{r}_\theta|$

To evaluate an integral in spherical coordinates, we express the region in spherical coordinates, substitute $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$ and $z = \rho \cos(\phi)$ in the function and include the integration factor $\rho^2 \sin(\phi)$.