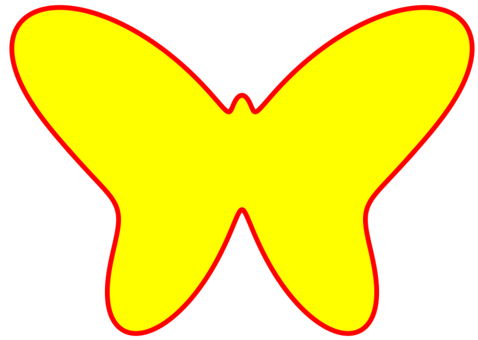
Homework 20: Polar integration

This homework is due Friday, 10/27 rsp Tuesday 10/31.

1 Find the area of the butterfly region

$$\iint_R 1 \, dA \; ,$$



where R is given in polar coordinates as $0 \leq r \leq r(\theta)$ where $r(\theta)$ is defined below.

$r(\theta) = 8 - \sin(\theta) + 2\sin(3\theta) + 2\sin(5\theta) - \sin(7\theta) + 3\cos(2\theta) - 2\cos(4\theta)$

2 Evaluate the given integral by changing to polar coordinates:

$$\iint_R 7x \, dA \; ,$$

where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

3 Use polar coordinates to find the volume of the solid bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

4 a) A city near the sea is modeled by a half disk $D = \{(x, y) \mid x^2 + y^2 \leq 49, x \geq 0$ with center the origin and radius 7. What is the average distance of a point in D to the origin? in other words, what is the integral $\int\int_D \sqrt{x^2 + y^2} \, dxdy / \int\int_D 1 \, dxdy$.
b) The distance to the beach is x. Find the average distance $\int\int_D x \, dxdy / \int\int_D 1 \, dxdy$ to the beach.

5 Evaluate the iterated integral

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} 9\sqrt{x^2 + y^2} \, dy \, dx \; .$$

Main definitions

Polar coordinates $(x, y) = (r\cos(t), r\sin(t))$ allow to describe regions bound by polar curves $(r(\theta), \theta)$.

The **average** of a quantity $f(x, y)$ over a region G is the fraction

$$\frac{\int\int_G f(x, y)\, dA}{\int\int_G 1\, dA} .$$

To integrate in polar coordinates, we evaluate the integral

$$\int\int_R f(x, y)\, dxdy = \int\int_R f(r\cos(\theta), r\sin(\theta))r\, drd\theta ,$$

where R is described in polar coordinates.

Example:

To integrate $f(x, y) = x^2 + y^2$ over the region $x^2 + y^2 \leq 9, x \geq 0, y \geq 0$, we integrate

$$\int_0^{\pi/2} \int_0^3 r^2 \cdot r\, drd\theta = (\pi/2)(3^4/4) = 81\pi/8 .$$