## Homework 17: Lagrange multipliers

This homework is due Friday, 10/20 rsp Tuesday 10/24.

**1** We look at a melon shaped candy. The outer radius is $x$, the inner is $y$. Assume we want to extremize the **sweetness function** $f(x, y) = -x^2 + 2y^2$ under the constraint that $g(x, y) = x - y = 2$. Since this problem is so tasty, we require you to use the most yummy method known to mankind: the **Lagrange** method! Is your solution a minimum or maximum?
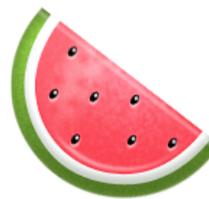
**2** The material to build a traffic light is $g(x, y) = 6 + 6\pi xy + 3\pi x^2 = 12$ is fixed (the radius of each cylinder is $x$ and the height is $y$ and the constant 6 is the material for the back plate). We want to build a light for which the shaded region with volume $f(x, y) = 3\pi x^2 y$ is maximal. Use the Lagrange method.
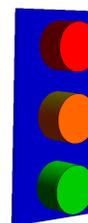
**3** The situation is the same, but we have two Lagrange multipliers (see box to the right). Use Lagrange multipliers to find the maximum and minimum $f$ under the two constraints:

$$f(x, y, z) = 3x - y - 3z;$$

$$g(x, y, z) = x + y - z = 0$$

$$h(x, y, z) = x^2 + 2z^2 = 1 .$$

**4** a) Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter 2 is equilateral.
b) What are the minima? Why does the Lagrange method not establish them?
*Hint:* Use Heron's formula $A = \sqrt{s(s-x)(s-y)(s-z)}$, where $s = 1$ and $x, y, z$ are the lengths of the sides.

**5** Which pyramid of height $h$ over a square $[-a, a] \times [-a, a]$ with surface area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$ has maximal volume $V(h, a) = 4ha^2/3$? By using new variables $(x, y)$ and multiplying $V$ with a constant, we get to the equivalent problem to maximize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$. Use the later variables.

## Main definitions

The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ for the three unknowns $x, y, \lambda$ are the **Lagrange equations**. $\lambda$ is a **Lagrange multiplier**. The **two constraint case** appears only here in homework and is not covered in section $\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0$ are the **Lagrange equations** $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z), \quad g(x, y, z) = 0, h(x, y, z) = 0$ are the **Lagrange equations** with two constraints.

**Lagrange theorem:** Maxima or minima of $f$ on the constraint $g = c$ are either solutions of the Lagrange equations or critical points of $g$.