Homework 14: Tangent lines and planes

This homework is due Friday, 10/13 rsp Thursday 10/12.

1 a) Find the tangent plane to the surface

$$x^2 + y^2 - x^2y^2 - z^2 = 0$$

at the point $(x, y, z) = (1, 2, 1)$.
b) Find the tangent line to the curve

$$x^2 + y^2 - x^2y^2 = -23$$

at the point $(x, y) = (3, 2)$.

2 In each of the following four conceptional problems, please answer briefly.
a) The figure 8 curve $f(x, y) = x^4 - x^2 + y^2 = 0$ has no tangent line at $(0, 0)$. Why? Isn't it perfectly smooth function $f(x, y)$?
b) The following statement is nonsense: "the tangent plane to the graph of $f(x, y, z) = x^2 + y^2 + z^2$ at $(x, y, z) = (1, 2, 3)$ is $2x + 4y + 6z = 28$". Modify it to make it a true statement.
c) Let $\vec{r}(u, v) = \langle u, v, g(u, v)\rangle$ and $\langle a, b, c\rangle = \vec{r}_u(1, 1) \times \vec{r}_v(1, 1)$. Let $f(x, y, z) = z - g(x, y)$ and $\langle A, B, C\rangle = \nabla f(1, 1, g(1, 1))\rangle$. What is the relation between $\langle a, b, c\rangle$ and $\langle A, B, C\rangle$?
d) Given a closed curve $x^4 + y^8 = 2$, there is a point (x_0, y_0) where the gradient is parallel to $\langle 3/5, 4/5\rangle$. Are there two points?

3 a) Find an equation of the tangent plane to the parametric surface

$$\vec{r}(u, v) = \langle u^2, v^2, uv\rangle$$

at the point $(u, v) = (1, 1)$.
b) The surface satisfies the equation $xy - z^2 = 0$. Find the tangent plane to this surface at the same point $(x, y, z) = (1, 1, 1)$ by computing the gradient.

4 Find an equation of the tangent plane and the normal line to the surface $x - z - 4\arctan(yz) = 0$ through the point $(1+\pi, 1, 1)$.

5 a) Show that the ellipsoid $6x^2 + 4y^2 + 2z^2 = 18$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$, meaning that they have the same tangent plane at that point.
b) Find a surface different from a plane for which $x + y + 2z = 4$ is the tangent plane at the point $(1, 1, 1)$.

Main definitions

The **gradient** in two dimensions is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$. In three dimensions, it is $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$.

From the chain rule, we can deduce:

Theorem: Gradients are orthogonal to level curves and level surfaces.

The tangent line through (x_0, y_0) to a level curve $f(x, y) = c$ is $ax + by = d$, where $\nabla f(x_0, y_0) = \langle a, b\rangle$ and d is obtained by plugging in the point. The tangent plane through (x_0, y_0, z_0) to a level surface $f(x, y, z) = C$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = \langle a, b, c\rangle$ and d is obtained by plugging in the point.

For parametrized surfaces $\vec{r}(u, v)$, the tangent plane is computed using the vectors $\vec{r}_u, \vec{r}_v$ are velocity vectors of grid curves and so tangent to the surface. The normal is $\vec{n} = \vec{r}_u \times \vec{r}_v = \langle a, b, c\rangle$ and then get $ax + by + cz = d$, where d is obtained by plugging in the point $\vec{r}(u_0, v_0)$.

Bonus problem

This problem is optional. If you submit this problem on Canvas until Friday night, we delete an other, least HW score. Change parameters, colors and curves to make your own art work, both in two dimensions as well as in three dimensions. This is not the Mathematica project we talked about. The Mathematica project at the end will however, like this little one be on the creative side. You can copy paste the following example code from the website. Start with that and modify until you get something you like. Then submit the text entry

```
L=1.2; r1=Thickness[0.01]; r2=Thickness[0.001]; c[t_]:=Hue[t/(2Pi)];
r[t_]:={Cos[t]+0.1*Sin[3t],Sin[t]+0.1*Sin[5t]};
F[t_]:=ParametricPlot[r[t]+s*r'[t],{s,-L,L},PlotStyle->{r2,c[t]}];
S1=ParametricPlot[r[t],{t,0,2Pi},PlotStyle->{White,r1}];
SS=Show[{S1,Table[F[t],{t,0,2Pi,0.01}]},Axes->False,AspectRatio->1,
  PlotRange->{{-2,2},{-2,2}},Background->RGBColor[0,0,0.2]]

L=1.1; r1=0.05; r2=0.02; n=100; c[t_]:=RGBColor[t/(2Pi),1-t/(2Pi),1];
r[t_]:={Cos[t]+0.1*Sin[3t],Sin[t]+0.1*Sin[5t], Cos[3t]+0.1*Sin[4t]};
F[t_]:=Graphics3D[{c[t],Tube[{r[t]-L*r'[t],r[t]+L*r'[t]},r2]}];
S1=Graphics3D[{White,Tube[Table[r[t],{t,0,2Pi,0.01}],r1]}];
TT=Show[{S1,Table[F[t],{t,0,2Pi,2Pi/n}]},Axes->False,AspectRatio->1,
PlotRange->All,Background->RGBColor[0.1,0,0],Boxed->False]

Export["art.png",GraphicsGrid[{{SS},{TT}}],"PNG",ImageSize->1200];
```