Homework 13: Chain Rule

This homework is due Wednesday, 10/12 (Monday 10/10 is Columbus day and no class), rsp Tuesday 10/11.

1 a) Use the chain rule to find the derivative $df(\vec{r}(t))/dt$ at $t = 1$ for

$$f(x, y) = \sin(x^5 + y^2) \ ,$$

where $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t^4, 1/t\rangle$. To do so, compute $r'(1)$ and $\nabla f(\vec{r}(1))$ and then the dot product. b) Now compute the derivative directly by differentiating $\sin((t^4)^5 + 1/t^2)$ with respect to t. You should get the same thing.

2 Find $dy/dx = y'(x)$ if x, y are related by

$$\sin(x) + \cos(y) = \sin(x)\cos(y) \ .$$

3 Find z_x and z_y for $yz = \log(x + z)$, where $\log = \ln$ is the natural log.

4 The radius of a right circular cone is increasing at a rate of 1.8 while its height is decreasing at a rate of 2.5. At what rate is the volume of the cone changing when the radius is 120 and the height is 140.

5 The Voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increase as the resistor heats up. Use **Ohm's Law**, $V = IR$, to find how the current I is changing at the moment when $R = 400$, $I = 0.08$ $dV/dt = -0.01$ and $dR/dt = 0.03$.

Main definitions

Define the **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$.

The **multivariable chain rule** is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\ .$$

When written out in two dimensions, this is

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

Example: a bug walks on $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$ on a plane with temperature $f(x,y) = x^2 + 5y$. Find the temperature change $d/dt f(\vec{r}(t))$ at $(1,0)$. **Solution:** either compose $f(\vec{r}(t)) = \cos^2(t) + 5\sin(t)$ and differentiate at $t = 0$ to get $d/dt f(\vec{r}(t)) = 5\cos(0) = 5$. Or then find $\vec{r}\,'(0) = \langle 0, 1\rangle$ and the gradient $\nabla f(x,y) = \langle 2x, 5\rangle$ which is $\langle 0, 5\rangle$ at $(1,0)$. The chain rule assures that the dot product is the same.
We can use the chain rule for implicit differentiation

Implicit differentiation: If $f(x,y) = c$ is a curve, we can compute $y' = -f_x/f_y$.

In three dimensions, the **implicit differentiation formulas** derived from the chain rule are:

$$z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$$

$$z_y(x,y) = -f_y(x,y,z)/f_z(x,y,z)$$