Homework 12: Linearization

This homework is due Friday, 10/6 rsp Tuesday 10/10.

1 a) Estimate $100000003^{1/4}$ without calculator by linearising $f(x) = x^{1/4}$ at $x = 100000000$. Compare with the actual value by using 30 digit accuracy. You have to use a tool like Mathematica. The command $N[100000003^{(1/4)}, 30]$ gives you the numerical value with 10 digits. b) Find the linearization $L(x, y)$ of the function $f(x, y) = x^3y^8$, at the point $(1, 1)$. Compare $L(1.01, 0.999)$ with $f(1.01, 0.999)$.

2 Find the linear approximation $L(x, y)$ of the function

$$f(x, y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2, 1)$ and use it to estimate $f(1.95, 1.04)$.

3 You know that the linearization of $f(x, y) = \sqrt{y + \cos^2 x}$ at a point (x_0, y_0) is $1 + \frac{1}{2}y$. Find (x_0, y_0).

4 Find the linear approximation $L(x, y)$ of the function $f(x, y) = \ln(x - 3y)$ at $(7, 2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing $z = f(x, y)$ and the plane $z = L(x, y)$, which we will later call tangent plane.

5 If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of $f(x, y) - f(x_0, y_0)$ and $L(x, y) - L(x_0, y_0)$. (The later value would be called a differential, but we do not use this expression).

Main definitions:

The **linear approximation** of a function $f(x)$ at a point a is the linear function

$$L(x) = f(a) + f'(a)(x - a) \ .$$

Example: Because $f(x) = \sqrt{x}$ has at the point $x_0 = 100$ the linearization $L(x) = f(x_0) + f'(x_0)(x - x_0) = 10 + (x - x_0)/20$, we can estimate $f(103) = \sqrt{103}$ as $L(103) = 10 + 3/20 = 10.15$ which is pretty close to the real value 10.1489.

The **linear approximation** of $f(x, y)$ at (a, b) is the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \ .$$

Example. Take $f(x, y) = \sqrt{x^3 y}$ and linearize near $(x_0, y_0) = (3, 3)$. We have $f_x(x, y) = 3x^2 y/(2\sqrt{x^3 y})$ and $f_y(x, y) = x^3/(2\sqrt{x^3 y})$ so that $f_x(3, 3) = 9/2$ and $f_y(3, 3) = 3/2$. linearization $L(x, y) = 9 + (9/2)(x - 3) + (3/2)(y - 3)$. We can estimate $\sqrt{2.999^3 * 3.00002}$ as $9 + (9/2)(-0.001) + (3/2)0.00002 = 8.99553$, which is $3.6 * 10^{-7}$ close to the real value.

The **linear approximation** of a function $f(x, y, z)$ at (a, b, c) is $L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$.

You might see the term "Differentials" in this context. It informally refers to the value of $L(x, y) - L(x_0, y_0)$. While tangent lines and tangent planes are level curves or level surfaces of L. We will have a special lecture on tangent planes where we compute them more efficiently.