Homework 11: Partial differential equations

This homework is due Wednesday, 10/4 rsp Thursday 10/5.

1 a) The following functions solve either the **Laplace equation** $u_{xx} + u_{yy} = 0$ or the **wave equation** $u_{xx} - u_{yy} = 0$. Decide in each case. Possible answers are "none", "both", "Wave equation" or "Laplace equation". As usual $\log = \ln$ is the natural log.

a) $u = 2x^2 + 2y^2$
b) $u = x^3 + 3xy^2$
c) $u = \log \sqrt{x^2 + y^2}$
d) $u = e^{-x} \cos y - e^{-y} \cos x$
e) $u = \sin(5x) \sin(5y)$
f) $u = \left(\frac{y}{y^2 - x^2}\right)$
g) $u = x^4 - 6x^2y^2 + y^4$
h) $u = \sin(x - y) + \log(x^2 + y^2)$

2 The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is an example of the **infamous Black-Scholes equation**. Here $f(x,t)$ is the price of a call option and x the stock price and t is time.
a) Verify that $f(t,x) = x$, $f(t,x) = (1+x^2)/(2x)$ and $f(t,x) = e^t$ solve this PDE. b) Verify that $e^t \log(x)$ solves Black-Scholes.
c) Verify that $e^{-3t}x^2$ solves Black Scholes.

3 a) Show that the **Cobb-Douglas** production function $P = L^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P .$$

The constants α and β are fixed. L is labor and K is capital.
b) Verify that $f(x,y) = \sqrt{x^2 + y^2}$ satisfies the **Eikonal equation** $f_x^2 + f_y^2 = 1$. This partial differential equation is important in optics. We will later see that it can be written as $|\nabla f| = 1$.

4 Run the Mathematica code for the two-dimensional wave equation. The example code is given on the website and below. Copy paste from the website, not from the PDF file. Plot the graph of the function $u(t, 0.3, 0.4)$ from $t = 0$ to $t = 1$. You can either print out your output or copy what you see on the screen.

5 The partial differential equation

$$f_t + f f_x = f_{xx}$$

called **Burgers equation** describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which is used to describe the weather. Use Mathematica to verify that the function

$$f(t,x) = \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

solves the Burgers equation.

An equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. You have to be able to verify whether some function solves a partial differential equation. You will also have to know 5 basic differential equations: the Laplace, the heat, the wave and the transport and the Burger equation. In lecture we provide some background on these equations and meaning which helps to know them without blindly memorizing them.

Use of Mathematica

Most of you have already installed Mathematica. If you have not done so, then part of the homework is also to finally install and run this program. You can use this software to make tough computations. You can use the software to do the problems above. Here is an example. After entering, type return while holding down the shift key:

```
f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[ D[f[t,x],t] == D[f[t,x],{x,2}]]
```

You have verified that the function

$$\frac{1}{\sqrt{t}}e^{-x^2/(4t)}$$

satisfies the heat equation. As any real programming language, Mathematica is particular about syntax. Watch brackets, capitalization, double equal signs ==! In general, square brackets are used in arguments of functions, round brackets are the usual brackets to organize formulas and curly brackets are lists

Here is Mathematica code for an example of the two dimensional wave equation. You can copy paste from the website. In the example, we see $f(x,y) = u(4,x,y)$ which is the wave at time 4. You will have to plot a graph of a function of t.

```
A=Rectangle[{0,0},{1,1}]; Clear[t,x,y];
f[x_,y_]:=Sin[2 Pi x] Abs[Sin[3 Pi y]];
g[x_,y_]:=2 Sin[Pi x] Sin[Pi 4 y];
U=NDSolveValue[{D[u[t,x,y],{t,2}]
 -Inactive[Laplacian][u[t,x,y],{x,y}]==0,
   u[0,x,y] == f[x,y],
   Derivative[1,0,0][u][0,x,y]==g[x,y],
   DirichletCondition[u[t,x,y] ==0,True]},
   u,{t, 0, 2 Pi}, {x,y} \[Element] A];
Plot3D[U[4,x,y],{x,0,1},{y,0,1}]
Manipulate[ContourPlot[U[t,x,y],
     {x,0,1},{y,0,1}],{t,0,2Pi}]
```