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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T F The functions $e^{x^2+y^3-y}$ and x^2+y^3-y have the same critical points.

Solution:
The gradient of the first function is $e^{x^2+y^3-y}$ times the gradient of the second.

2) T F The line $\vec{r}(t) = \langle t^2, t^2, t^2 \rangle$ hits the plane $x+y+z=100$ at a right angle.

Solution:
Yes, the normal vector to the plane is parallel to $\langle 1,1,1 \rangle$.

3) T F The quadric $x^2 - 2y^2 + z^2 = 5$ is a one sheeted hyperboloid.

Solution:
It is stretched a bit in the y direction by a factor $\sqrt{2}$ and scaled by a factor $\sqrt{5}$ but otherwise, it is a standard hyperboloid

4) T F The relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ is only possible if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector.

Solution:
It is also possible if they are nonzero but form a 45 degree angle.

5) T F The partial differential equation $u_x = u_{tt}$ is called the **Heat equation**.

Solution:
The variables are reversed.

6) T F The curvature of the curve $\vec{r}(t) = \langle \sin(2t), \cos(2t)/\sqrt{2}, \cos(2t)/\sqrt{2} \rangle$ at $t=0$ is equal to the curvature of the curve $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$ at $t=0$.

Solution:
Both are circles of radius 1 which have curvature 1.

7) T F The space curve $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$ for $t \in [0, 10\pi]$ is located on a cylinder.

Solution:
We can check $x^2 + z^2 = 1$.

8) T F If a smooth function $f(x, y)$ has a global maximum, then it has a global minimum.

Solution:
A counter example is $f(x, y) = -x^2 - y^2$.

9) T F If $L(x, y)$ is the linearization of $f(x, y)$ at (x_0, y_0) and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ on $f = c$ at the point $\vec{r}(t_0) = \vec{s}(t_0) = (x_0, y_0)$ so that $|\vec{r}\,'(t_0)| = |\vec{s}\,'(t_0)| = 1$, then $|d/dtL(\vec{s}(t))| = |d/dtf(\vec{r}(t))|$ at the time $t = t_0$.

Solution:
Locally, near $(x - 0, y_0)$ the linearization $L(x, y)$ is close to the function $f(x, y)$. The level surface to f through (x_0, y_0) and the level surface of $L(x, y)$ have the same normal vector $\vec{n} = \nabla f = \nabla L$. The vectors $\vec{v} = \vec{s}\,'(t_0)$ and $\vec{w} = \vec{r}\,'(t_0)$ are both normal to the same vector and so parallel. Since they have the same length, either $\vec{v} == \vec{w}$ or $\vec{v} = -\vec{w}$. By the chain rule $d/dtL(\vec{s}(t)) = \vec{n} \cdot \vec{v} = \pm \vec{n} \cdot \vec{w} = \pm d/dtf(\vec{r}(t))$. The absolute values agree.

10) T F If $\vec{F}$ is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot \vec{dr}$ is either positive or zero.

Solution:
The power $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$ is positive and so is the integral.

11) T F The flux of the vector field $\vec{F} = \nabla f$ through the surface $f(x, y, z) = x^4 + y^4 + z^4 = 1$ is positive if the surface is oriented so that $\vec{r}_u \times \vec{r}_v$ points in the direction of the gradient of f.

Solution:
If we parametrize the surface with $\vec{r}(u, v)$, then the flux is $\int\int_S \nabla f(\vec{r}(u, v)) \cdot (r_u \times r_v)\, dudv$. The integrand is positive by assumption.

12) T F If we extremize the function $f(x, y)$ under the constraint $g(x, y) = 1$, and the functions are the same $f = g$, all points on the constraint curve are extrema for f.

Solution:
Every point on the curve $g(x, y) = 1$ is a solution to the Lagrange equations because $\nabla f = \nabla g$.

13) | T | **F** |

If a point (x_0, y_0) is a minimum of $f(x, y)$ under the constraint $g(x, y) = 1$, then it is also a local minimum of the function $f(x, y)$ without constraints.

Solution:
The gradient of f does not have to be the zero vector.

14) | **T** | F |

If a vector field $\vec{F}(x, y)$ is a gradient field, then any line integral along any closed ellipse is zero.

Solution:
This follows from the fundamental theorem of line integrals.

15) | T | **F** |

The flux of an irrotational vector field is zero through any surface S in space.

Solution:
First of all, the surface does not need to be closed. But even if the surface is closed, this would still be false: the field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ is irrotational but the flux through any closed surface is zero.

16) | T | **F** |

The divergence of a gradient field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is everywhere zero.

Solution:
While the curl of gradient is zero and the divergence of a curl is zero, the divergence of a gradient is the Laplacian of f and not necessarily zero.

17) | **T** | F |

The line integral of the vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ along a circle in the $xy-$ plane is zero.

Solution:
Yes, by the fundamental theorem of line integrals.

18) T **F** For any solid E, the moment of inertia $\iiint_E x^2 + y^2 \; dxdydz$ is always larger than the volume $\iiint_E 1 \; dxdydz$ of E.

Solution:
For small solids, the moment of inertia is small, for large solids, the moment of inertia is large.

19) **T** F The curvature of a parametrized curve satisfying $|\vec{r}\,'(t)| = 1$ is bounded above by the length $|\vec{r}\,''|$ of the acceleration.

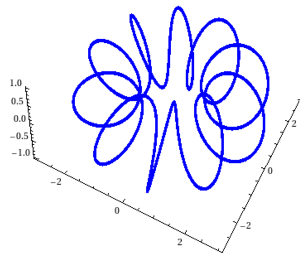
Solution:
One can deduce this directly from the formula for curvature $|\vec{r}\,' \times \vec{r}\,''| \leq |\vec{r}\,''|$ in that case. Actually, more is true, the curvature is in this case the length of the acceleration.

20) **T** F Given a vector field $\vec{F} = \langle P, Q, R \rangle$, the directional derivative of $\text{div}(\vec{F}(x,y,z))$ in the direction $\vec{v} = \langle 1, 0, 0 \rangle$ is $P_{xx} + Q_{xy} + R_{xz}$.

Solution:
Yes by definition $\text{div}(\vec{F}(x,y,z)) = P_x(x,y,z) + Q_y(x,y,z) + R_z(x,y,z)$. The directional derivative in the $\langle 1, 0, 0 \rangle$ direction is the partial derivative with respect to x. We use also Clairot.

Problem 2) (6 points)

a) (6 points) Match the objects with their definitions



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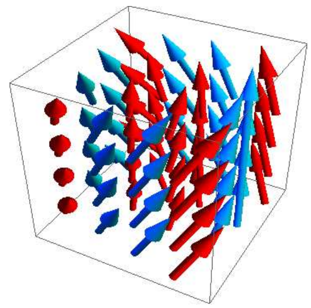

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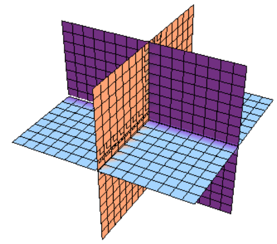

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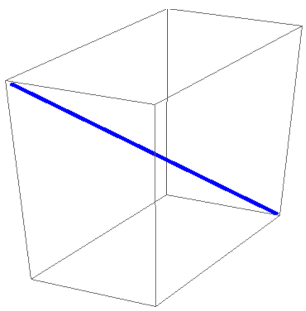

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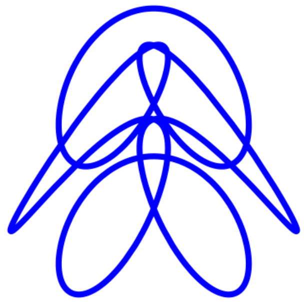

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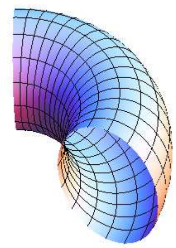
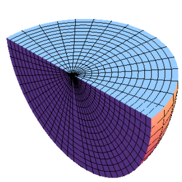


6

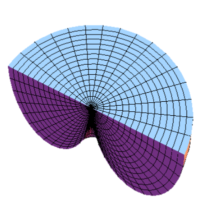
Enter 1-6	Object definition
	$\vec{r}(t) = \langle (2+\cos(10t))\cos(t), (2+\cos(10t))\sin(t), \sin(10t)\rangle$
	$\vec{F}(x,y,z) = \langle -y, x, 2\rangle$
	$\vec{r}(t,s) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s)\rangle$
	$x^2y^2z^2 = 0$
	$(x-1)/5 = (y-2)/10 = (z-1)/3$
	$\vec{r}(t) = \langle \sin(t)+\cos(5t), \cos(t)+\cos(6t)\rangle$

b) (4 points) Match the solids with the triple integrals:

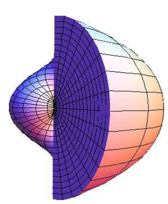
Enter A-D	3D integral computing volume
	$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{1/\cos(\phi)} \rho^2\sin(\phi)\ d\rho d\phi d\theta$
	$\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{\sin(\phi)} \rho^2\sin(\phi)\ d\rho d\phi d\theta$
	$\int_0^{\pi}\int_{\pi/2}^{\pi}\int_0^{1} \rho^2\sin(\phi)\ d\rho d\phi d\theta$
	$\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi-\theta} \rho^2\sin(\phi)\ d\rho d\phi d\theta$



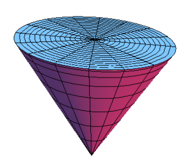
A



B



C

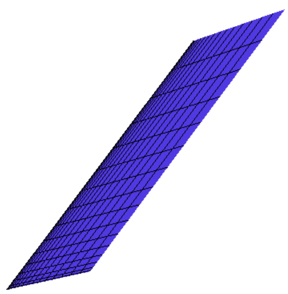


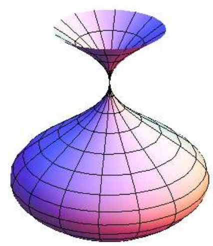
D

Solution:
a) 1,2,6,3,4,5
b) D,B,A,C

Problem 3) (10 points)

a) (6 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.


A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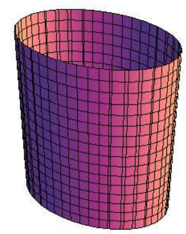
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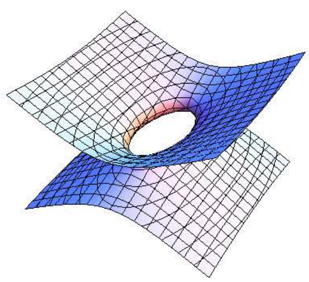
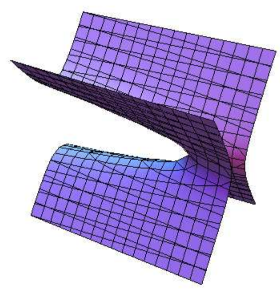
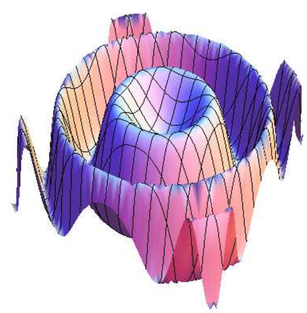

D


E


F

Enter A-F here	Function or parametrization
	$\vec{r}(u,v) = \langle u^2, v^2, u^2+v^2 \rangle$
	$\vec{r}(u,v) = \langle (1+\sin(u))\cos(v), (1+\sin(u))\sin(v), u \rangle$
	$4x^2 + y^2 - 9z^2 = 1$
	$x - 9y^2 + 4z^2 = 1$
	$\vec{r}(u,v) = \langle u, v, \sin(u^2+v^2) \rangle$
	$4x^2 + 9y^2 = 1$

b) (4 points) If the blank box is replaced by $\nabla f(5,6)$ the statement becomes true or false. Determine which case we have. The function $f(x,y)$ is an arbitrary nice function like for example $f(x,y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x,y) = f(5,6)$ and has the property that $\vec{r}(0) = \langle 5, 6 \rangle$.

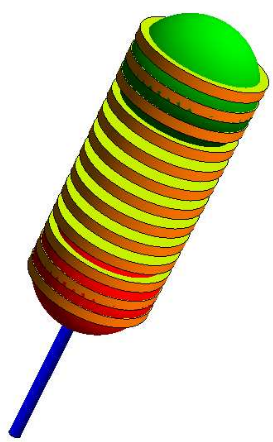
True/False	Topic	Statement
	Linearization	$L(x,y) = f(5,6) + \boxed{} \cdot \langle x-5, y-6 \rangle$
	Chain rule	$\frac{d}{dt} f(\vec{r}(t))\|_{t=0} = \boxed{} \cdot \vec{r}\,'(0)$
	Steepest descent	f decreases at $(5,6)$ most in the direction of $\boxed{}$
	Estimation	$f(5+0.1, 5.99) \sim f(5,6) + \boxed{} \cdot \langle 0.1, -0.01 \rangle$
	Directional derivative	$D_{\vec{v}} f(5,6) = \boxed{} \cdot \vec{v}$, $\|\vec{v}\| = 1$
	Level curve	of f through $(5,6)$ has the form $\boxed{} \cdot \langle x-5, y-6 \rangle = 0$
	Vector projection	of $\nabla f(5,6)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \boxed{})/\|\vec{v}\|^2$
	Tangent line	of $\vec{r}(t)$ at $(5,6)$ is parametrized by $\vec{R}(s) = \langle 5,6 \rangle + s \boxed{}$

Solution:

a) A,B,D,E,F,C

b) T,T,F,T,T,F,T,F

Problem 4) (10 points)



Two ice cream scoops given by spheres

$$x^2 + y^2 + (z+1)^2 = 1$$

and

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = 1$$

are enclosed by a cylinder which is tangent to both spheres. Find the equation of the cylinder.

Hint: consider the distance of a general point (x, y, z) to the line passing through the centers of the spheres.

Solution:
The line L passes through the points $A = (0, 0, -1)$ and $B = (1, 1, 2)$ which are connected by the vector $\vec{v} = \langle 1, 1, 3 \rangle$. The distance from a point $P = (x, y, z)$ to the line is given by

$$d(P, L) = \frac{|\langle x - 0, y - 0, z + 1 \rangle \times \langle 1, 1, -3 \rangle|}{|\langle 1, 1, -3 \rangle|} .$$

The cross product in that formula

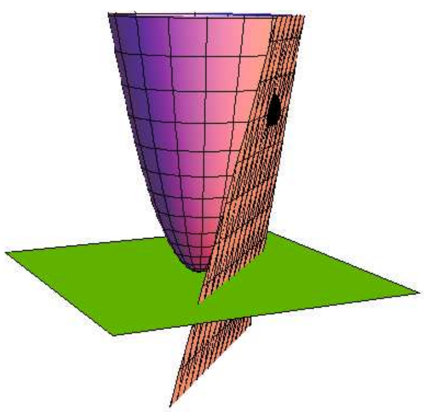
$$\langle x, y, z + 1 \rangle \times \langle 1, 1, 3 \rangle = \langle 3y - z - 1, z + 1 - 3x, x - y \rangle .$$

The equation $d(P, L) = 1$ is equivalent to $d(P, L)^2 = 1$ which is

$$(3y - z - 1)^2 + (z + 1 - 3x)^2 + (x - y)^2 = 11 .$$

Its fine to leave the result like this but one could write it as $\boxed{-6x + 10x^2 - 6y - 2xy + 10y^2 + 4z - 6xz - 6yz + 2z^2 = 9}$.

Problem 5) (10 points)



Find a parametrization

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

for the line obtained by intersecting the tangent plane Σ to the surface

$$x^2 + y^2 - z = 0$$

at $(-1, -1, 2)$ with the xy-plane.

Solution:
The gradient of f is $\nabla f(x,y,z) = \langle 2x, 2y, -1\rangle$ and at the given point $\nabla f(-1,-1,2) = \langle -2,-2,-1\rangle$. The tangent plane is

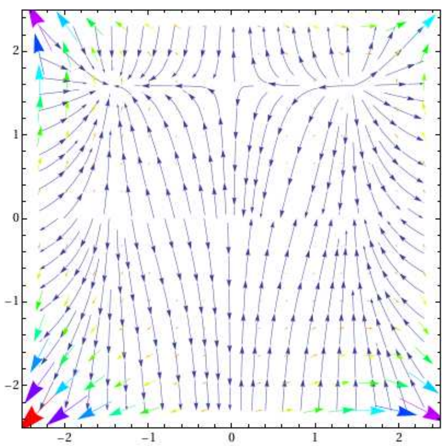
$$-2x - 2y - z = 2 \ .$$

For $z = 0$, we have $x + y = -1, z = 0$. While these two equations describe the line, we were asked to compute a parametrization. It is most conveniently found by picking two points on the line like $(-1,0,0)$ and $(0,-1,0)$. The vector connecting these points is $\langle 1,-1,0\rangle$. The parametrization of the line is

$$\vec{r}(t) = \langle -1,0,0\rangle + t\langle 1,-1,0\rangle \ .$$

The answer is $\boxed{\vec{r}(t) = \langle t-1, -t, 0\rangle}$.

Problem 6) (10 points)



The vector field

$$\vec{F}(x,y) = \langle P, Q\rangle = \langle y(x^4 - 2x^2), x(y^4 - 4y)\rangle$$

has the curl

$$f(x,y) = \text{curl}(\vec{F})(x,y) = Q_x(x,y) - P_y(x,y) \ .$$

Find and classify all critical points of f by deciding whether they are local maxima, local minima or saddle points. Is there a global maximum or global minimum of f?

Solution:
We have $Q_x = y^4 - 4y, P_y = x^4 - 2x^2$ so that we have to find the extrema of the function

$$f(x,y) = y^4 - 4y - x^4 + 2x^2 .$$

The gradient of f is

$$\nabla f(x,y) = \langle -4x^3 + 4x, 4y^3 - 4 \rangle .$$

The critical points satisfy the two equations

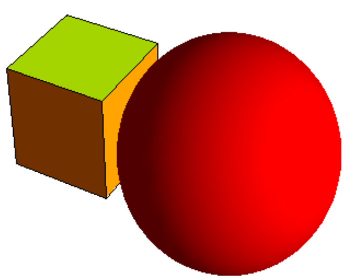
$$-4x^3 + 4x = 0$$
$$4y^3 - 4 = 0 .$$

This leads to $y = 1$ and $x = -1, 0, 1$ so that we have three critical points. To apply the **second derivative test**, we have to compute $f_{xx} = -12x^2 + 4$ and the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = (-12x^2 + 4)(12y^2)$.

Point	D	f_{xx}	Nature of critical point	local or global
$(-1,1)$	-96	-8	saddle	neither max nor min
$(0,1)$	48	4	min	local minimum
$(1,1)$	-96	-8	saddle	neither max nor min

There are no global maxima nor global minima because the function takes arbitrarily large negative values $x = 0, x \to -\infty$ and arbitrarily large values $x = 0, y \to \infty$.

Problem 7) (10 points)



We want to minimize the volume of the union of a **sphere** of radius x and a **cube** of side length y under the constraint that the sum of the two surface areas is equal to 4. Find the minimal value using the Lagrange method.

Remark: You do not have to show any derivations of the volume and surface area of the sphere.

Solution:
We want to minimize the function

$$f(x,y) = 4\pi x^3/3 + y^3$$

under the constraint

$$g(x,y) = 4\pi x^2 + 6y^2 = 4 .$$

The Lagrange equations are

$$\begin{array}{rcl} 4\pi x^2 & = & \lambda 8\pi x \\ 3y^2 & & \lambda 12 y \\ 4\pi x^2 + 6y^2 & = & 4 . \end{array}$$

These equations have a solution with $x = 0$ which is $(x,y) = (0, \sqrt{2/3})$ and a solution with $y = 0$ which is $(x,y) = (\sqrt{1/\pi}, 0)$ and a solution, where both x and y are nonzero. To get this solution, divide the first equation by the second, to get $(4/3)\pi x^2/y^2 = (2/3)\pi x/y$ or $2x = y$. This gives $x = (6+\pi)^{-1/2}$ and $y = 2(6+\pi)^{-1/2}$. This point is the minimum. The answer is $\boxed{(x,y) = ((6+\pi)^{-1/2}, 2(6+\pi)^{-1/2})}$.

Remark: this problem is inspired by the corresponding two dimensional problem presented in the planar exam review. The 2D problem in the plane has been described in the new book "The Mathematical Mechanic: Using Physical Reasoning to Solve Problems by Mark Levi, 2009". The book shows how the solution can be explained using physical reasoning. Both in the two and three dimensional situation, the cube length is the same than the diameter of the circle or sphere.

Remark. Some more work had been required here, if the formula for the surface area of the sphere were not known. In that case, one had to do the computation $\int_0^{2\pi} \int_0^{\pi} R^2 \sin(\phi)\ d\phi d\theta = 4\pi R^2$ and $\int_0^{2\Pi} \int_0^{\pi} \int_0^R \rho^2 \sin(\phi) d\rho d\phi d\theta = 4\pi R^3/3$.

Problem 8) (10 points)

A solid E in space is determined by the inequalities

$$0 \leq z \leq 9 ,$$

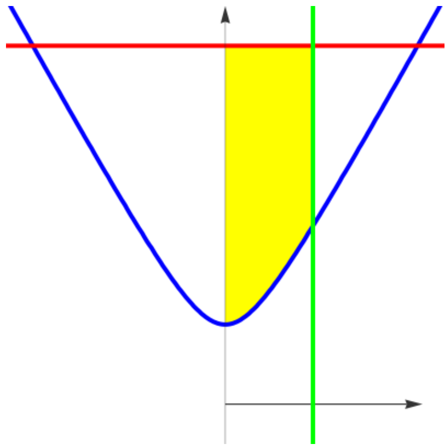
$$z^2 - x^2 - y^2 \geq 4$$

and

$$x^2 + y^2 \leq 1 .$$

Find the volume of E.

Solution:

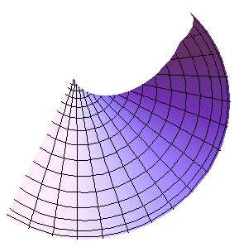


Use **cylindrical coordinates**. It is crucial to **draw the situation** in the rz-plane to read off the integration bounds: the region is bounded from below by the hyperboloid $z^2 - r^2 = 4$ (a hyperbola in the rz-plane) from above by the plane $z = 9$ (a horizontal line in the rz plane) and to the side by the cylinder $r = 1$ (a vertical line in the rz-plane). The computation is:

$$\int_0^{2\pi} \int_0^1 \int_{\sqrt{4+r^2}}^9 r dz dr d\theta = 2\pi \int_0^1 r(9 - \sqrt{4+r^2})\, dr = \pi(9 + 16/3 - 2 \cdot 5^{3/2}/3) .$$

The final result is $\boxed{\pi(9 + 16/3 - 2 \cdot 5^{3/2}/3)}$.

Problem 9) (10 points)



A surface S is parametrized by

$$\vec{r}(u,v) = e^{-u^2} \langle 1, \sin(v), \cos(v) \rangle$$

where

$$0 \leq u \leq \sqrt{\pi}, u^2 \leq v \leq \pi .$$

Find its surface area.

Solution:
We have $|r_u \times r_v| = \sqrt{8} u e^{-2u^2}$. The integral

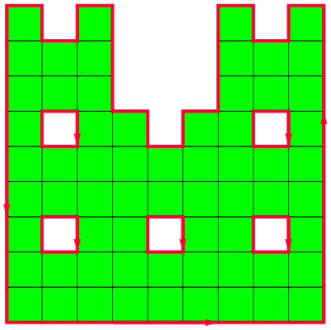
$$\int_0^{\sqrt{\pi}} \int_{u^2}^{\pi} \sqrt{8} u e^{-2u^2} dv du$$

becomes after a **switch of variables** (make a picture!)

$$\int_0^{\pi} \int_0^{\sqrt{v}} \sqrt{8} u e^{-2u^2} du dv .$$

The result is $\boxed{(2\pi - 1 + e^{-2\pi})/(2\sqrt{2})}$.

Problem 10) (10 points)



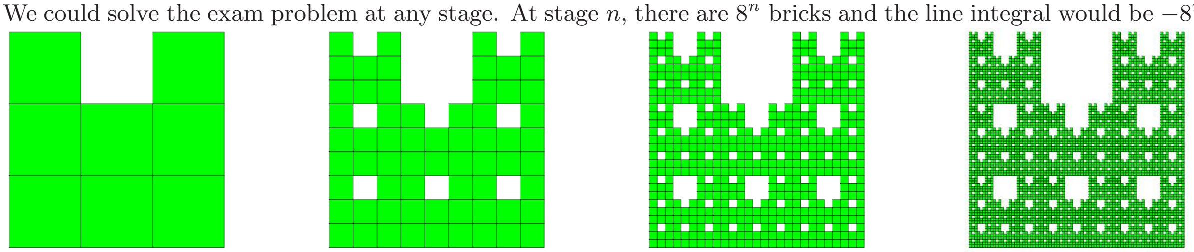
What is the line integral $\int_C \vec{F} \cdot \vec{dr}$ of the vector field

$$\vec{F}(x, y) = \langle 1 + y + 2xy, y^2 + x^2 \rangle$$

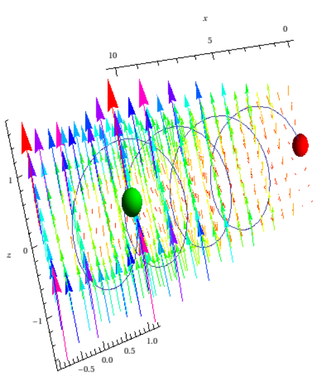
along the boundary C of the planar "castle region" shown in the picture? Each of the 5 windows is a unit square and the base of the castle has length 9. The boundary consists of 6 curves which are all oriented so that the region is to the left.

Solution:
The curl $Q_x - P_y$ of the vector field is constant -1. The line integral is already given in the correct orientation so that it is by **Green's theorem** equal to the double integral of the curl over the region G. Since the castle has $81 - 9 - 8 = 64$ "bricks", the result is $-area(G) = \boxed{-64}$.

The castle by the way is a second step to a fractal construction. You can stack 8 such castles on each other to get a new one, then rescale. In the limit, we get a **fractal** a self similar structure, where one 1/8'th of the castle is similar to the castle itself. Because dividing things up into 3 parts leads to 8 times more bricks, the dimension is fractional $1 < \log(8)/\log(3) < 2$, hence the name fractal. [The dimensionality formula appears in the Star treck movie, when young spock learns math.] Without the hole, it would be a 2D object because $\log(9)/\log(3) = 2$. We could solve the exam problem at any stage. At stage n, there are 8^n bricks and the line integral would be -8^n.



Problem 11) (10 points)



Compute the line integral of the vector field

$$\vec{F}(x, y, z) = \langle \cos(x), 2 + \cos(y), e^z + x(y^2 + z^2) \rangle$$

along the curve $\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ with $0 \le t \le 3\pi$.

Hint: you might want to find a split $\vec{F} = \vec{G} + \vec{H}$ and compute line integrals of $\vec{G}$ and $\vec{H}$ separately.

Solution:
The vector field is the sum of a gradient field $\vec{G} = \langle \cos(x), 2 + \cos(y), e^z \rangle = \nabla f$ with $f(x,y,z) = \sin(x) + 2y + \sin(y) + e^z$ and the rest $\vec{H} = \langle 0, 0, x(y^2+z^2) \rangle$. We have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r} + \int_C \vec{H} \cdot d\vec{r} \ .$$

The first line integral is by the **fundamental theorem of line integrals** $f(3\pi, -1, 0) - f(0,1,0) = -4 - 2\sin(1)$. The second line integral can be computed directly (note that $y^2 + z^2 = 1$ on the curve)

$$\begin{aligned} \int_0^{3\pi} \vec{F}(\vec{r}(t) \cdot \vec{r}\,'(t)\ dt &= \int_0^{3\pi} \langle 0, 0, t \rangle \cdot \langle 1, -\sin(t), \cos(t) \rangle\ dt \\ &= \int_0^{3\pi} t\cos(t)\ dt \\ &= t\sin(t)|_0^{2\pi} - \int_0^{3\pi} \sin(t)\ dt \\ &= 0 + \cos(t)|_0^{3\pi} = -2 \ . \end{aligned}$$

The sum of the two integrals is $\boxed{-6 - 2\sin(1)}$.

Problem 12) (10 points)

A biker in the Harvard Hemenway gym pedals. Assume that the force of a foot is

$$\vec{F} = \langle 0, 0, x^3 - x^2 + \sqrt{2 + \sin(z)} \rangle$$

and that one of the feet moves on a path $C : \vec{r}(t) = \langle 2\cos(t), 0, 2\sin(t) \rangle$. How much work

$$\int_C \vec{F} \cdot d\vec{r}$$

is done by this foot, when pedaling 10 times which means $0 \le t \le 20\pi$?

Solution:
By **Stokes theorem**, the line integral from 0 to 2π is equal to the flux of the curl of $\vec{F}$ through any surface which has C as a boundary. We take the simplest surface S which the disc parametrized by $\vec{r}(u,v) = \langle u, 0, v\rangle$ which has the normal vector $\vec{r}_u \times \vec{r}_v = \langle 1,0,0\rangle \cdot \langle 0,0,1\rangle = \langle 0,-1,0\rangle$. The curl of the vector field is

$$\text{curl}(\vec{F})(x,y,z) = \langle 0, -3x^2 + 2x, 0\rangle \ .$$

The flux of the curl is is

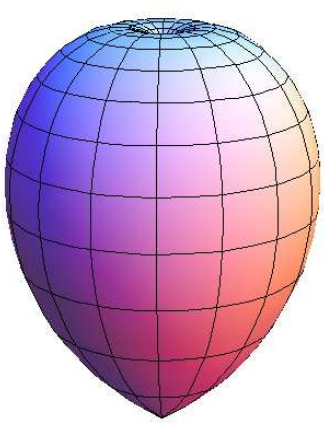
$$\int\int_{u^2+v^2 \leq 4} \langle 0, -3u^2 + 2u, 0\rangle \cdot \langle 0,-1,0\rangle \ dudv = \int\int_{u^2+v^2\leq 4} 3u^2 - 2u \ dudv \ .$$

This double integral is best computed in polar coordinates $u = r\cos(\theta), v = r\sin(\theta)$ leading to

$$\int_0^{2\pi} \int_0^2 [3r^2\cos^2(\theta) - 2r\cos(\theta)]r \ drd\theta \ = \pi(3 \cdot 2^4/4) = 12\pi \ .$$

Peddaling 10 rounds leads to the work $\boxed{120\pi}$.

Problem 13) (10 points)



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x,y,z) = \langle z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y)\rangle \ .$$

A **tonsil** is given in spherical coordinates as $\rho \leq \phi$. Find the flux of the X-Ray field $\vec{F}$ through the surface $\rho = \phi$ of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.

Solution:
The divergence of the vector field is constant 2. The flux through the tonsil surface is therefore by the divergence theorem 2 times the volume of the tonsil. This volume integral is best done in **spherical coordinates**:

$$2\int_0^{2\pi}\int_0^{\pi}\int_0^{\phi}\rho^2\sin(\phi)\;d\rho d\phi d\theta = (4\pi/3)\int_0^{\pi}\phi^3\sin(\phi)\;d\phi\;.$$

This integral needs integration by parts:

$$\begin{aligned}
2\;\text{Volume(tonsil)} &= \frac{4\pi}{3}[-\phi^3\cos(\phi)|_0^{\pi} + \int_0^{\pi} 3\phi^2\cos(\phi)\;d\phi] \\
&= \frac{4\pi}{3}[\pi^3 - 3\int_0^{\pi}\sin(\phi)2\phi\;d\phi] \\
&= \frac{4\pi}{3}[\pi^3 + 6\phi\cos(\phi)_0^{\pi} - 6\int_0^{\pi}\cos(\phi)\;d\phi] \\
&= \frac{4\pi}{3}(\pi^3 - 6\pi)\;.
\end{aligned}$$

The computation could be simplified a bit by switching the order of integration (for the triangle in the ϕ, ρ plane) requiring less integration by parts. The result is still $\boxed{\frac{4\pi}{3}(\pi^3 - 6\pi)}$.