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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

1) T **F** The angle between the vectors $\langle 1,2,3\rangle$ and $\langle 2,-1,-1\rangle$ is $\frac{\pi}{6}$.

Solution:
The dot product is negative. It is an obtuse angle and can not be 30°.

2) **T** F $\langle 4,6,8\rangle$ is a normal vector for the plane $-2x-3y-4z=5$.

Solution:
Yes, it is parallel to the gradient vector $\langle -2,-3,-4\rangle$ of the plane.

3) T **F** The plane tangent to the graph of $f(x,y)=x^2+y^2$ at $(3,4,25)$ is $6x+8y=50$.

Solution:
The equation describes an elliptic paraboloid, not a circle in the plane. We are in space, not in the two dimensional plane. The answer given is the equation of the tangent line to the circle of radius 5.

4) **T** F The directional derivative of a function f in the direction of ∇f can never be negative.

Solution:
$D_{\nabla f} f = \nabla f \cdot \nabla f \geq 0$.

5) **T** F The surface parameterized by $\langle \sin u, \cos v, u^2+v^2\rangle$, $0 \leq u,v \leq 1$ has the same surface area as the surface parameterized by $\langle \sin u^2, \cos v^2, u^4+v^4\rangle$, $0 \leq u,v \leq 1$.

Solution:
Yes, surface area does not depend on the parametrization.

6) **T** F By the chain rule, $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\; dt = f(\vec{r}(b)) - f(\vec{r}(a))$.

Solution:
Yes, this is the proof of the fundamental theorem of line integrals.

7) | T | F | The function $u(x,y) = x^2 + y^2$ is a solution of the wave equation $u_{xx} = u_{yy}$.

Solution:
Yes, $u_{xx} = 2$ and $u_{yy} = 2$.

8) | T | F | Let C be the unit circle parametrized counter-clockwise. If $\vec{F}(x,y)$ is a vector field and $\int_C \vec{F} \cdot d\vec{r} = 0$, then $\vec{F}$ is a gradient vector field.

Solution:
It is not sufficient to check the closed loop property for one closed path only. It has to be true for all closed paths.

9) | T | F | The vector field $\vec{F}(x,y,z) = \langle y^2 - z^2, z^2 - x^2, x^2 - y^2 \rangle$ is conservative.

Solution:
It is incompressible because the divergence is zero but it is not conservative, because the curl is not identically zero. Already the third component is not.

10) | T | F | The vector field $\vec{F}(x,y,z) = \langle x,y,z \rangle$ is the curl of a vector field.

Solution:
If it were of the form $\vec{G} = \text{curl}(\vec{F})$ then the divergence would have to be zero.

11) | T | F | Let $\vec{F}(x,y) = \langle x^2, y^2 \rangle$ and $\vec{G}(x,y) = \langle x^2 - y, y^2 + x \rangle$. If C is the unit circle, traveled counter-clockwise, then $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}$.

Solution:
The second vector field is the sum of $\vec{F}$ and a vector field $\langle -y, x \rangle$ which has constant curl 2. The second integral is by π larger than the first.

12) | T | F | If $\vec{F}(x,y,z) = \langle 1,1,1 \rangle$, and C is a curve, then $\int_C \vec{F} \cdot d\vec{r}$ is the arc length of C.

Solution:
If we write down the line integral we get $\int_a^b P(r(t)) + Q(r(t)) + R(r(t))\ dt$ and not $\int_a^b \sqrt{P(r(t))^2 + Q(r(t))^2 + R(r(t))^2}\ dt$.

13) | T | F | The function $f(x,y,z) = e^z$ is the divergence of a vector field $\vec{F}(x,y,z)$.

Solution:
Yes, it is the divergence for example of $\vec{F}(x,y,z) = \langle 0,0,e^z \rangle$.

14) | T | F | Given a function $f(x,y) = x^3 + xy^2$ in the plane, then the flow lines of grad(f) are perpendicular to the level curves of f.

Solution:
Yes, the flow lines are parallel to the gradient and the gradient is perpendicular to the level curves.

15) | T | F | Given a function $f(x,y)$ without critical points, and a curve $\vec{r}(t)$ which is always perpendicular to grad(f), then $\vec{r}(t)$ is a piece of level curve of f.

Solution:
Being perpendicular to the gradient is parallel to the level curve.

16) | T | F | The curl of a conservative vector field in space may not be conservative.

Solution:
The curl of a conservative vector field is the zero vector field.

17) | T | F | If a vector field $\vec{F}$ in space is incompressible (div$(\vec{F}) = 0$) and irrotational (curl$(\vec{F}) = \vec{0}$), it is necessarily constant.

Solution:
A counter example is $\vec{F} = (x, -y, 0)$.

18) | T | **F** |

If a particle moves along a circle, its acceleration vector always points to the center of the circle.

Solution:
This is true for a constant speed but not in general. We gave also full credit if someone said yes, and assumed the speed is constant.

19) | **T** | F |

$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{-x^2-y^2-z^2} \, dz \, dy \, dx = \int_0^1 \int_0^{1-z} \int_0^{1-z-y} e^{-x^2-y^2-z^2} \, dx \, dy \, dz$.

Solution:
Make a picture

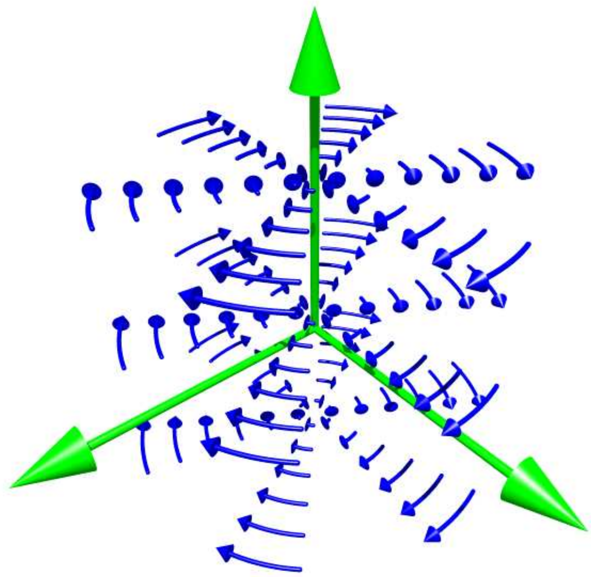
20) | **T** | F |

$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 1 \, dz \, dy \, dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin(\phi) \; d\rho \, d\phi \, d\theta$.

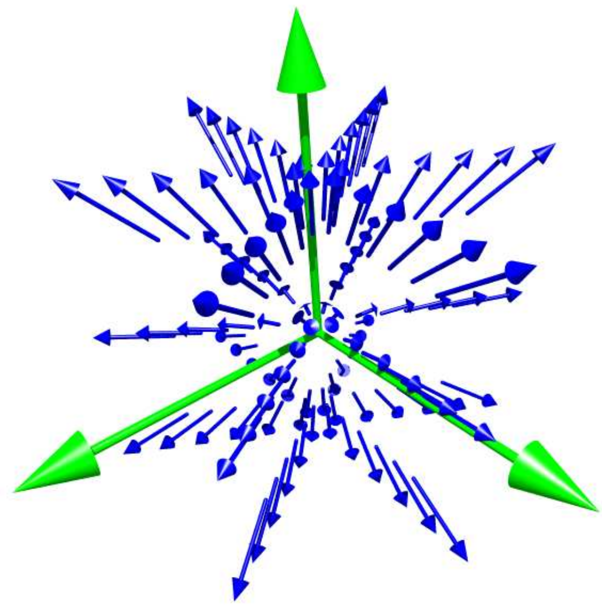
Solution:
This is the integral over an octant intersected with a solid sphere. .

Problem 2) (10 points)

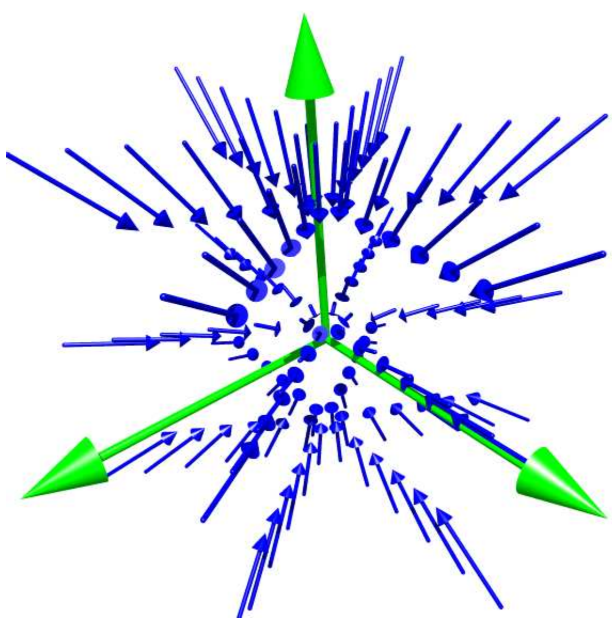
No justifications are needed in this problem. Match the following vector fields in space with the corresponding formulas:



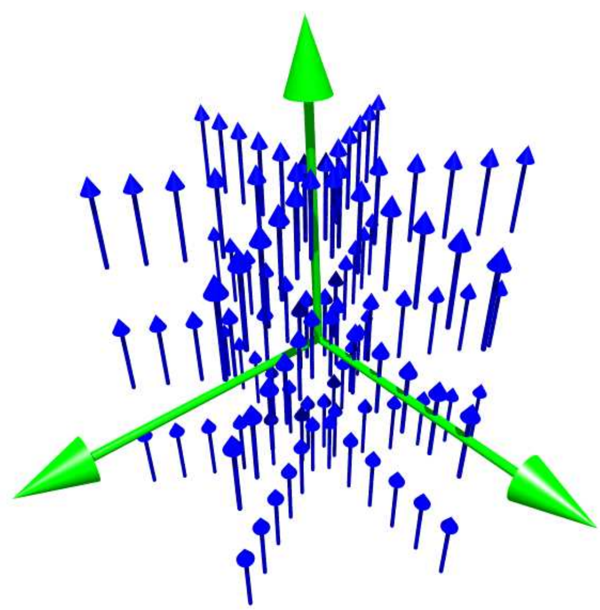
I



II



III



IV

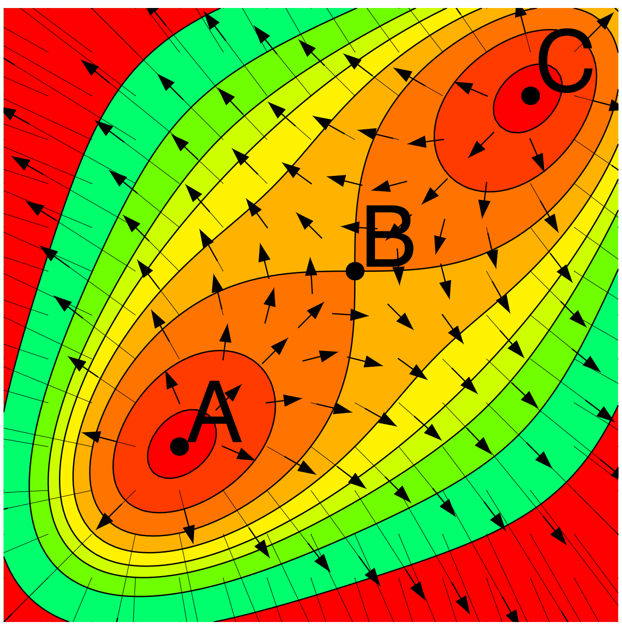
Enter I,II,III,IV here	Vector Field
	$\vec{F}(x,y,z) = \langle y, -x, 0\rangle$
	$\vec{F}(x,y,z) = \langle x, y, z\rangle$
	$\vec{F}(x,y,z) = \langle 0, 0, 1\rangle$
	$\vec{F}(x,y,z) = \langle -x, -2y, -z\rangle$

Solution:

Enter I,II,III,IV here	Vector Field
I	$\vec{F}(x,y,z) = \langle y, -x, 0\rangle$
II	$\vec{F}(x,y,z) = \langle x, y, z\rangle$
IV	$\vec{F}(x,y,z) = \langle 0, 0, 1\rangle$
III	$\vec{F}(x,y,z) = \langle -x, -2y, -z\rangle$

Problem 3) (10 points)

No justifications are required in this problem. The first picture shows a gradient vector field $\vec{F}(x,y) = \nabla f(x,y)$.



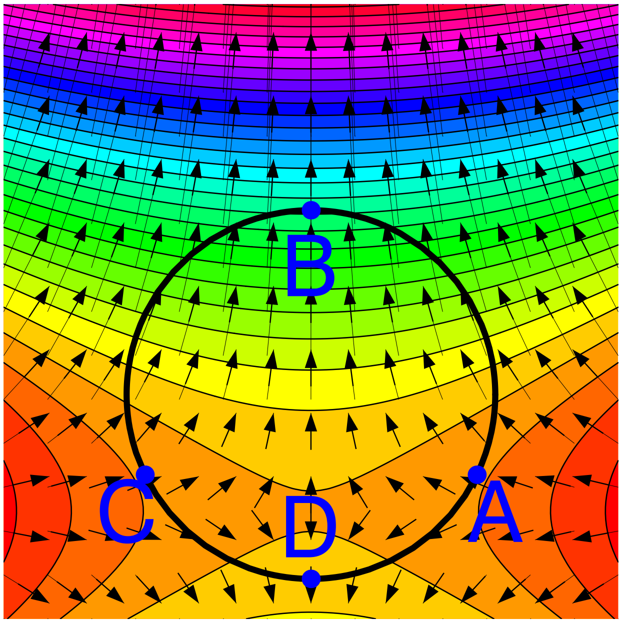
The critical points of $f(x,y)$ are called A, B and C. What can you say about the nature of these three critical points? Which one is a local max, which a local min, which a saddle.

point	local max	local min	saddle
A			
B			
C			

$\int_{P\to Q} \vec{F} \cdot \vec{dr}$ denotes the line integral of $\vec{F}$ along a straight line path from P to Q.

statement	True	False
$\int_{A\to B} \vec{F}\, \vec{dr} \geq 0$		
$\int_{A\to C} \vec{F}\, \vec{dr} \geq \int_{A\to B} \vec{F}\, \vec{dr}$		

The second picture again shows an other gradient vector field $\vec{F} = \nabla f(x,y)$ of a different function $f(x,y)$.



We want to identify the maximum of $f(x, y)$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$. The solutions of the Lagrange equations in this case are labeled A, B, C, D. At which point on the circle if f maximal?

point	maximum
A	
B	
C	
D	

$\int_\gamma \vec{F} \cdot \vec{dr}$ denotes the line integral of $\vec{F}$ along the circle $\gamma : x^2 + y^2 = 1$, oriented counter clockwise.

$\int_\gamma \vec{F} \cdot \vec{dr}$	> 0	< 0	$= 0$
Check if true:			

Solution:
It is important to realize in this problem that the gradient vector points into the direction where f increases. The points A, C are local minima, the point B is a saddle point. By the **fundamental theorem of line integrals**, the line integral of any path from A to B is $f(B) - f(A)$. This is ≥ 0. The line integral from A to C is $f(C) - f(A)$. This is smaller than the line integral from A to B. The point B is the global maximum of f in the second picture. The line integral of F along the circle is zero again by the fundamental theorem of line integrals. A gradient vector field has the property that the line integral along a closed loop is zero.

Problem 4) (10 points)

Given a tetrahedron with vertices $A = (-1, -1, -1)$, $B = (1, 0, 0)$, $C = (0, 1, 0)$ and $D = (0, 0, 1)$, find the distance between the edges AC and BD.

Solution:
$\vec{AB} = \langle 2, 1, 1 \rangle$, $\vec{AC} = \langle 1, 2, 1 \rangle$, $\vec{BD} = \langle -1, 0, 1 \rangle$, The distance is $|(\vec{AC} \times \vec{BD}) \cdot \vec{CD}|/|(\vec{AC} \times \vec{BD})| = \boxed{2/\sqrt{3}}$.

Problem 5) (10 points)

The vector field

$$\vec{F}(x, y) = \langle P, Q \rangle = \langle -\frac{x^5}{5} - 2y^2x, -4x^2\frac{y^3}{3} \rangle$$

has the divergence $f(x, y) = \mathrm{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$. Classify all the critical points of this function f.

Solution:
The function we want to extremize is

$$f(x, y) = \mathrm{div}(\vec{F})(x, y) = -x^4 - 2y^2 - 4x^2y^2 \ .$$

The gradient of f is $\nabla f(x, y) = \langle -4x(x^2 + 2y^2), -y4(1 + 2x^2) \rangle$. It has the only critical point $(0, 0)$. The discriminant D is zero at this point so that the second derivative test is **inconclusive**. We still can make a statement: from the formula, we see that $f(0, 0) = 0$ and that $f(x, y)$ is negative everywhere else. The point $(0, 0)$ must be a $\boxed{\text{global maximum}}$.

Problem 6) (10 points)

a) (4 points) Find the linearization $L(x, y)$ of the function $P(x, y) = x^2 - y$ and the linearization $K(x, y)$ of $Q(x, y) = \sin(\pi y) + x$ at the point $(0, 1)$.

b) (4 points) Estimate the **vector**

$$\vec{F}(0.01, 0.9999)$$

for the vector field

$$\vec{F}(x,y) = \langle x^2 - y, \sin(\pi y) + x \rangle = \langle M, N \rangle$$

using linear approximation.

c) (2 points) The vector field $\vec{G}(x,y) = \langle L(x,y), K(x,y) \rangle$ is called the **linearization** of the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$. Is the curl of $\vec{F}$ equal to the curl of $\vec{G}$ at $(0,1)$?

Solution:
a) The gradient of P is $(2x,-1)$ which is at the point $(0,1)$ equal to $(0,-1)$. The gradient of N is $(1, \pi\cos(\pi y))$ which is at the point $(0,1)$ equal to $(1,-\pi)$. The linearization of P at $(0,1)$ is $L(x,y) = P(0,1) + 0(x-0) + (-1)(y-1) = -1-(y-1) = -y$. The linearization of N at $(0,1)$ is $K(x,y) = Q(0,1) + x - \pi(y-1) = x - \pi(y-1)$. The new vector field is $\boxed{\vec{G}(x,y) = \langle -y, x - \pi(y-1) \rangle}$.
b) The estimation is given by $L(0.01, 0.9999) = (-1) + 0.01 \cdot 0 + 0.0001 = -0.9999$ and $K(0.01, 0.9999) = 0.01 - \pi(0.0001)$. We have $G(0,1) = F(0,1) = \langle -1, 0 \rangle$ and from $\vec{G}(0,01, 0.9999) = \langle -0.9999, 0.01 - \pi(0.9999-1) \rangle = \boxed{\langle -0.9999, 0.01 + 0.0001\pi \rangle}$.
c) $\boxed{\text{Yes}}$, the curl involves only the first derivatives of $\vec{F}$ or $\vec{G}$ and these agree at $(0,1)$.

Problem 7) (10 points)

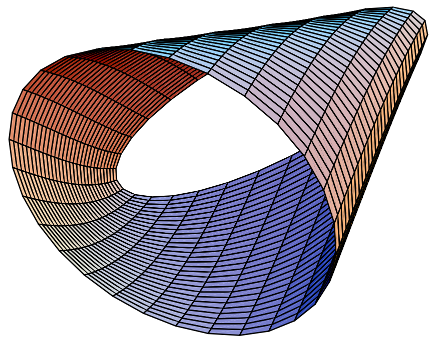
The parametrized surface

$$\vec{r}(u,v) = \langle \cos(u), v, \sin(u+v) \rangle$$

for $0 \leq u \leq 2\pi, 0 \leq v \leq \pi/3$ has a boundary which consists of two distinct curves C_1, C_2.

a) (5 points) Write down the integral for the surface area of S. You do not have to evaluate the integral in this problem a).

b) (5 points) Parameterize both boundaries and compute the arc length of one of the two boundary curves. The orientation of the parameterizations is not important. Of course, you have to find the value of the integral in this part b). It is possible to find it for one of the two boundary curves.



Solution:
a) We compute $\vec{r}_u = \langle -\sin(u), 0, \cos(u+v)\rangle$ and $\vec{r}_v = \langle 0, 1, \cos(u+v)\rangle$ and $\vec{r}_u \times \vec{r}_v = \langle -\cos(u+v), \sin(u)\cos(u+v), -\sin(u)$. Therefore $|\vec{r}_u \times \vec{r}_v| = \sqrt{\cos^2(u+v)(1+\sin^2(u)) + \sin^2(u)}$ and

$$Area = \int_0^{2\pi} \int_0^{\pi/3} |\vec{r}_u \times \vec{r}_v| \, dvdu \; .$$

b) The boundaries of the surface are obtained when taking $v = 0$ or $v = \pi/3$. The first case leads to a circle $\vec{r}(u) = \langle \cos(u), 0, \sin(u)\rangle$. The arc length of this circle is $\boxed{2\pi}$

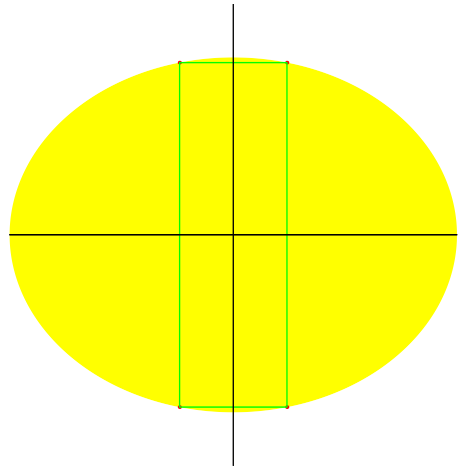
Problem 8) (10 points)

Which rectangle inscribed into the planar region

$$x^4 + y^8 = 17$$

has maximal area?

Note. As the picture indicates, the rectangle is symmetric both with respect to the x-axes and the y-axes and has all its sides parallel to the x axes or y axes.



Solution:
We extremize the function $f(x,y) = 4xy$, which is the area of the rectangle. The Lagrange equations are

$$\begin{aligned} 4y &= \lambda 4x^3 \\ 4x &= \lambda 8y^7 \\ x^4 + y^8 &= 17 \end{aligned}$$

Dividing the first equation by the second gives $x/y = 2y^7/x^3$ or $x^4 = 2y^8$. Plug this into the third equation to get $3y^8 = 17$ or $y = (17/3)^{1/8}$ and $x = (34/3)^{1/4}$.

Problem 9) (10 points)

Evaluate the double integral

$$\int_0^2 \int_{x^2}^4 \frac{x}{e^{y^2}} \; dydx \; .$$

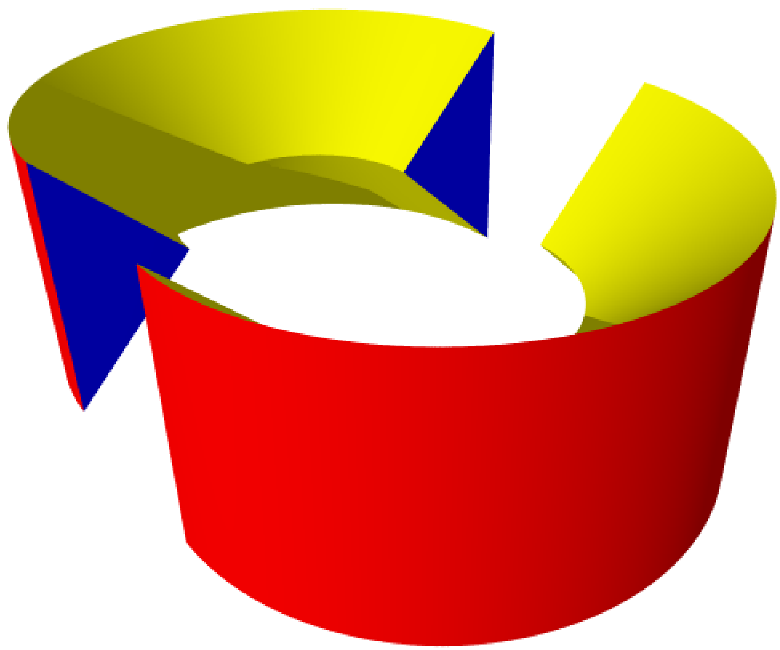
Solution:
Change the order of integration:

$$\int_0^4 \int_0^{\sqrt{y}} \frac{x}{e^{y^2}} \; dxdy = -\frac{1}{4} e^{-y^2}|_0^4 = \frac{1-e^{-16}}{4} \; .$$

It can also be written as $\boxed{(e^{16}-1)/(4e^{16})}$.

Problem 10) (10 points)

A solid is made by intersecting the solid cylinder $x^2 + y^2 \leq 4$ with $x^2 + y^2 \geq (z+1)^2$ and $x^2 + y^2 \geq (z-1)^2$. Find the volume of this body. The picture shows the body sliced along the xz-plane to show the crosssection. It actually is in one piece.



Solution:
The solid intersects with a fixed plane $\theta = 0$ in a triangle in the rz plane. This triangle is visible in the cross section already. We integrate in the r direction from 1 to 2 and get for each r the bounds $(1-r)$ and $(r-1)$. The answer is

$$\int_0^{2\pi} \int_1^2 \int_{1-r}^{r-1} r dz dr d\theta = 2\pi \int_1^2 (2r-2) r \, dr = 10\pi/3 \, .$$

Problem 11) (10 points)

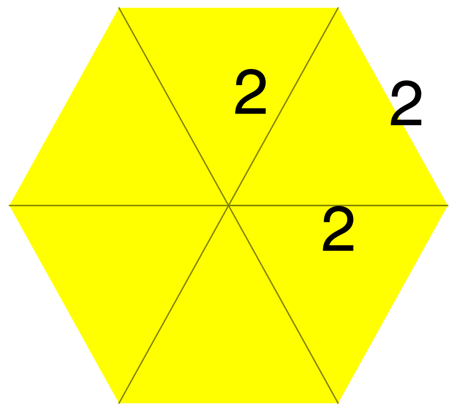
Compute the line integral of the vector field

$$\vec{F}(x,y) = \langle 0, x + e^{\sin(e^y)} \rangle$$

along the boundary of a regular hexagon with vertices

$$(2,0), (1,\sqrt{3}), (-1,\sqrt{3}), (-2,0), (-1,-\sqrt{3}), (1,-\sqrt{3}) .$$

Hint. As the picture indicates, the hexagon is the union of 6 identical equilateral triangles with side length 2.



Solution:
The curl of $\vec{F}$ is equal to 1. The line integral is the area of the hexagon, which is $\boxed{6\sqrt{3}}$.

Problem 12) (10 points)

Evaluate the line integral

$$\int_C \langle y^3, 3xy^2 + e^z, ye^z \rangle \cdot d\vec{r}$$

where C is the curve parameterized by $\vec{r}(t) = \langle e^t, t^5, \log(t^{10} - t^5 + 1) \rangle$, and where the parameter t satisfies $0 \leq t \leq 1$.

Solution:
The vector field is a gradient field because $\text{curl}(\vec{F}) = \vec{0}$ for all (x,y,z). The potential function is $f(x,y,z) = xy^3 + ye^z$. By the fundamental theorem of line integrals the line integral is $f(\vec{r}(1)) - f(\vec{r}(0)) = f(e,1,0) - f(1,0,0) = e+1-0$. The final answer is $\boxed{e+1}$.

Problem 13) (10 points)

Evaluate the integral

$$\iint_S \operatorname{curl}(\langle y + e^{\cos(z)}, e^{\sin(z)}, ze^{e^x}\rangle) \cdot d\vec{S} \ ,$$

where S is the part of the paraboloid $16 - x^2 - y^2$ above the xy- plane with orientation so that the normal vector points **downward**.

Hint: Relate this integral to an integral over the disc $x^2 + y^2 \leq 16, z = 0$.

Solution:
By **Stokes theorem**, the flux integral is the same as the line integral along the boundary of S, which is a circle of radius 4 in the xy-plane oriented clockwise (because the surface is oriented downwards). This integral is the same as the integral through the surface which is the disc in the xy-plane oriented downwards. Because the curl of the vector field is

$$\operatorname{curl}(\vec{F})(x, y, z) = \langle -e^{\sin(z)} \cos(z), \sin(z) \left(-e^{\cos(z)}\right), -1\rangle$$

which is on the xy plane $z = 0$ equal to is $\langle -1, 0, -1\rangle$, the answer is just the area of the disc, which is $\boxed{16\pi}$.

Problem 14) (10 points)

Find the flux integral

$$\iint_S \vec{F} \cdot d\vec{S} \ ,$$

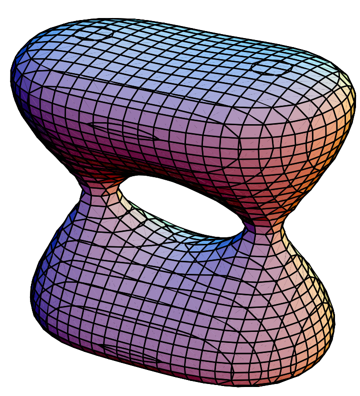
where $\vec{F}$ is the vector field

$$\vec{F}(x, y, z) = \langle y^2 - xz + e^y, -yz + z^{3x+6z^2}, x^4 + y^2 + z^2\rangle \ ,$$

and where S is the surface given by

$$f(x, y, z) = (x^2 + y^2)^2 - (x^2 - y^2) - z^2 + \frac{1}{5} + \frac{(x^6 + y^6 + z^6)}{10} = 0$$

with orientation pointing outward.



Solution:
By the **divergence theorem**, the flux integral over the closed surface is the integral

$$\int\int\int_E \text{div}(\vec{F}(x, y, z))\ dxdydz$$

Because $\text{div}(\vec{F}(x, y, z)) = 0$ for all points (x, y, z), this integral is zero. Consequently, also $\int\int_S \vec{F}\ d\vec{S} = 0$. The final answer is $\boxed{0}$.