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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

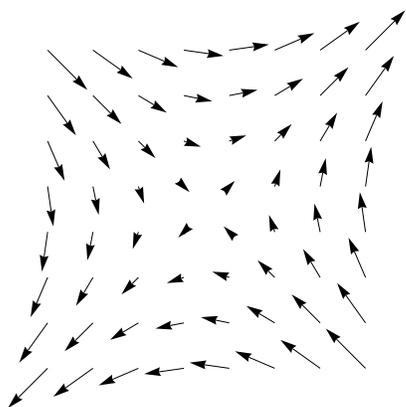
- 1)  T  F The function  $f(x, y, z) = x^2 - y^2 - z^2$  increases in the direction  $\langle -3, -1, 2 \rangle / \sqrt{14}$  at the point  $(1, 1, 1)$ .
- 2)  T  F The unit tangent vector of the curve  $\vec{r}(t) = \langle 3t, 4t, t^2 \rangle$  at time  $t = 0$  is  $\langle 3/5, 4/5, 0 \rangle$ .
- 3)  T  F There exist two nonzero vectors  $\vec{a}$  and  $\vec{b}$  such that the length of the vector projection of  $\vec{a}$  to  $\vec{a} \times \vec{b}$  is  $\frac{1}{2}|\vec{b}|$ .
- 4)  T  F The arc length of the curve  $\vec{r}_1(t) = \langle e^{3t^3} - 1, t^6 + 2, \sin(2t^3) \rangle$ ,  $0 \leq t \leq 1$  is larger than that of  $\vec{r}_2(t) = \langle e^{3t} - 1, t^2 + 2, \sin(2t) \rangle$ ,  $0 \leq t \leq 1$ .
- 5)  T  F The tangent plane of the graph of  $f(x, y) = \sin(x) + y^3$  at  $(0, 1, 1)$  is  $x + 3y = 3$ .
- 6)  T  F There exists a curve  $C$  on the level surface of  $f(x, y, z) = x^3 + e^{yz} + \cos(y) = 2$  such that the line integral  $\int_C \nabla f \cdot d\vec{r} > 0$ .
- 7)  T  F If  $Q$  is the point away from the plane  $3x + 5y + z = 7$  and  $P$  is the point on the plane closest to  $Q$ , then  $\vec{PQ}$  is parallel to  $\langle 3, 5, 1 \rangle$ .
- 8)  T  F The vector field  $\vec{F}(x, y, z) = \langle y^2 - xz + e^y, -yz, x^4 + y^2 - z^2 \rangle$  is the curl of a vector field  $\vec{G}$ .
- 9)  T  F Let  $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$  and  $C$  be the unit circle oriented counterclockwise. Since  $Q_x = P_y$  everywhere, Green implies  $\int_C \vec{F} \cdot d\vec{r} = 0$ .
- 10)  T  F By linear approximation of the function  $f(x, y, z) = e^{x+y+z}$  we can estimate  $f(0.1, 0.01, 0.001)$  as 1.111.
- 11)  T  F If  $\vec{F}(x, y, z)$  is a vector field defined on  $0 < x^2 + y^2 + z^2 < 4$  and  $\text{curl}(\vec{F}) = 0$  everywhere on this solid, then  $\vec{F} = \nabla f$  for some function  $f$ .
- 12)  T  F The tangent plane of the surface  $x^2 + y^4 + z^6 = 6$  at  $(2, 1, 1)$  is perpendicular to the line  $\vec{r}(t) = \langle 1 + 2t, 3 + 2t, -4 + 3t \rangle$ .
- 13)  T  F Given two curves  $C_1 : \vec{r}_1(t) = \langle t, t^3 \rangle$ ,  $0 \leq t \leq 1$  and  $C_2 : \vec{r}_2(s) = \langle s, s^5 \rangle$ ,  $0 \leq s \leq 1$   $f(x, y) = \sin(x^2y)$ . Then  $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ .
- 14)  T  F If  $f(x, y)$  has a global maximum, then the discriminant function  $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$  has a global maximum.
- 15)  T  F Let  $\vec{F}(x, y, z) = \langle x, y, z \rangle$  and  $S$  the surface boundary of the cube  $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$  oriented by outward normal vectors. Then  $\int \int_S \vec{F} \cdot d\vec{S} = 0$ .
- 16)  T  F Let  $\vec{F}(x, y, z) = \langle x/3, y/3, z/3 \rangle$  and  $S$  the unit sphere oriented by the outward normal vectors. Then  $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$  is the volume of the unit ball.
- 17)  T  F In three dimensional space there exist two nonzero vector fields  $\vec{F}$  and  $\vec{G}$  such that  $\text{curl}(\vec{F}) = \text{div}(\vec{G})$ .
- 18)  T  F The vector field  $\vec{F}(x, y, z) = \langle \cos(y), \cos(z), \cos(x) \rangle$  has the property that  $\vec{F} = \text{curl}(\text{curl}(\vec{F}))$ .
- 19)  T  F There exists a vector field  $\vec{F}(x, y, z)$  defined on  $\mathbf{R}^3$  such that every line integral  $\int_C \vec{F} \cdot d\vec{r}$  of  $\vec{F}$  over a closed curve  $C$  is equal to 0, but not every surface integral  $\int \int_S \vec{F} \cdot d\vec{S}$  over a closed surface  $S$  is equal to 0.
- 20)  T  F Whenever  $\vec{F} = \nabla f$ , for some function  $f(x, y)$  defined on the annulus  $\frac{1}{2} \leq x^2 + y^2 \leq 2$ , then  $\int_C \vec{F} \cdot d\vec{r} = 0$ , where  $C$  is the circle  $x^2 + y^2 = 1$ .

Problem 2) (10 points)

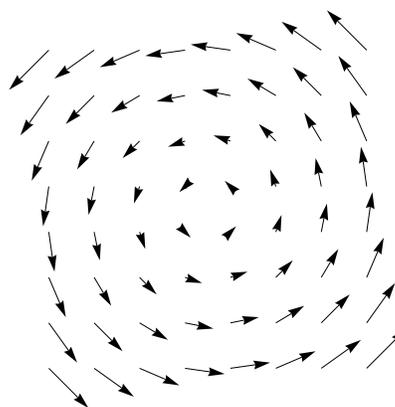
a) (5 points) We match in this problems vector fields with properties of vector fields and formulas for vector fields. A field  $\vec{F}$  is **divergence free** if  $\text{div}(\vec{F}) = 0$  everywhere in the plane. A field  $\vec{F}$  is **irrotational**, if  $\text{curl}(\vec{F}) = \vec{0}$  everywhere in the plane. In the last two columns of the following table, check the boxes which apply.

field	enter I-IV	divergence free	irrotational
$\vec{F}(x, y) = \langle -y, x \rangle$			
$\vec{F}(x, y) = \langle y, x \rangle$			
$\vec{F}(x, y) = \langle -x - y, x - y \rangle$			
$\vec{F}(x, y) = \langle x + y, x + y \rangle$			

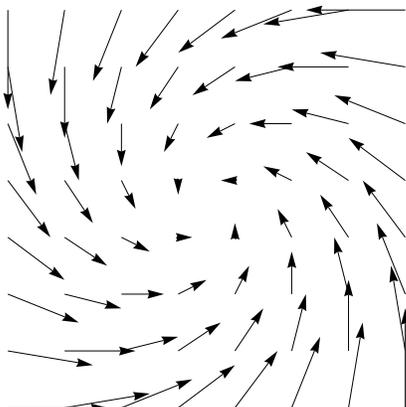
I



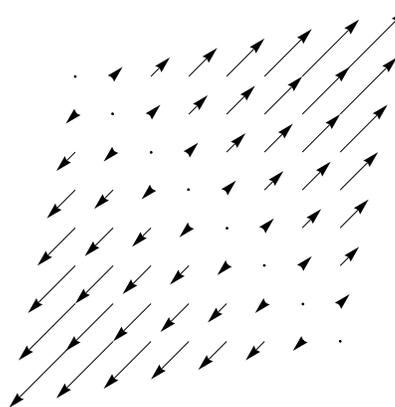
II



III



IV



b) (5 points) Match the following names of partial differential equations with functions  $u(t, x)$  which satisfy the differential equation and with formulas defining these equations.

equation	A-D	1-4
wave		
heat		
transport		
Laplace		

A	$u(t, x) = t^2 + x^2$
B	$u(t, x) = t^2 - x^2$
C	$u(t, x) = \sin(x + t)$
D	$u(t, x) = x^2 + 2t$

1	$u_t(t, x) = u_x(t, x)$
2	$u_{tt}(t, x) = u_{xx}(t, x)$
3	$u_{tt}(t, x) = -u_{xx}(t, x)$
4	$u_t(t, x) = u_{xx}(t, x)$

Problem 3) (10 points)

a) (6 points) Select 6 of the integrals  $A - H$  in the lower tables and match them with their names in the following table:

name	label A-H
line integral	
flux integral	
surface area	
arc length	
volume	
area	

$\iint_R x^2 - y^2 \, dx dy$	A
$\iint_R 1 \, dx dy$	B
$\iiint_R 1 \, dx dy dz$	C
$\iiint_R x^2 + z^2 \, dx dy dz$	D

$\iint_R \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v \, du dv$	E
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$	F
$\int_a^b  \vec{r}'(t)  \, dt$	G
$\iint_R  \vec{r}_u \times \vec{r}_v  \, du dv$	H

b) (4 points)

derivative	enter A-D
divergence	
curl	
gradient	
directional derivative	

The middle column of the following table is obtained by applying a derivative operation to the object in the left column. Fill in the correct label (A-D) of that operation into the above table.

object	derivative	label
$\vec{F}(x, y, z) = \langle -y, x, x \rangle$	$\langle 0, -1, 2 \rangle$	A
$\vec{F}(x, y, z) = \langle x^2, y, x \rangle$	$2x + 1$	B
$f(x, y, z) = x^2 + y^2 + z$	$\langle 2x, 2y, 1 \rangle$	C
$f(x, y, z) = x^3 + 5y^2$	$10y$	D

Problem 4) (10 points)

Consider the tetrahedron with vertices

$$A = (0, 1, -1), B = (4, 0, -1), C = (2, 1, 3), \text{ and } D = (2, 2, 0).$$

- a) (3 points) What is the area of the parallelogram spanned by  $\vec{AB}$  and  $\vec{AD}$ ?
- b) (3 points) Find the volume of the parallelepiped spanned by  $\vec{AC}$ ,  $\vec{AB}$  and  $\vec{AD}$ .
- c) (4 points) Determine the distance between the two skew lines  $AB$  and  $CD$ .

Problem 5) (10 points)

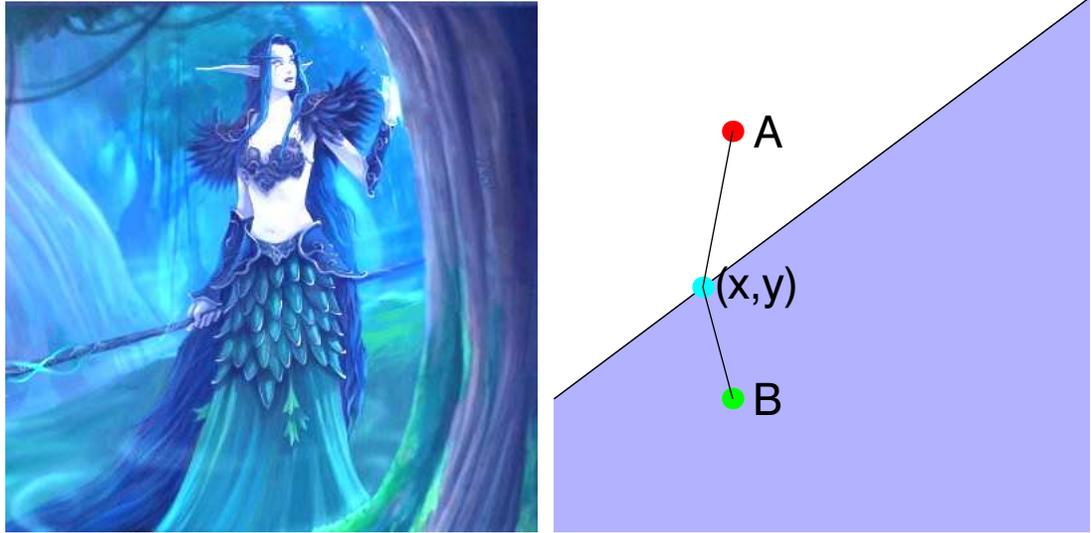
- a) (5 points) The curl of  $\vec{F}(x, y) = \langle -e^{xy}, y \rangle$  is equal to a scalar function  $f(x, y)$ . Estimate  $f(1.1, 0.001)$  by linear approximation.
- b) (5 points) Using the same function as in a), the equation  $f(x, y) = \text{curl}(\vec{F})(x, y) = 1$  defines  $y$  as a function  $g(x)$  of  $x$  near  $x = 1$ . Find  $g'(1)$ .

Problem 6) (10 points)

Find all the critical points of the function  $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$  and classify them by telling whether they are local maxima, local minima or saddle points.

Problem 7) (10 points)

A nightelf in the game World of Warcraft runs from  $A = (0, 2)$  to  $B = (0, 0)$  along a straight line segment from  $A$  to  $(x, y)$  and swims through the lake  $x - y \geq -1$  from  $(x, y)$  to a gold chest located at  $B = (0, 0)$  again on a straight line segment. The effort from  $A$  to  $(x, y)$  is the square of the distance from  $A$  to  $(x, y)$ . Her effort from  $(x, y)$  to  $B$  is 2 times the squared distance from  $(x, y)$  to  $B$ . Using the Lagrange method, find the choice of a drop point  $(x, y)$  on the lake shore that minimizes her effort.



Problem 8) (10 points)

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r},$$

where  $C$  is the curve given by

$$\vec{r}(t) = \left\langle \frac{t\pi}{2}, 1 - t, t^3 \right\rangle, 0 \leq t \leq 1$$

and

$$\vec{F}(x, y, z) = \langle e^{y^2} + z \cos(xz), 2xye^{y^2}, x \cos(xz) \rangle.$$

Problem (9) (10 points)

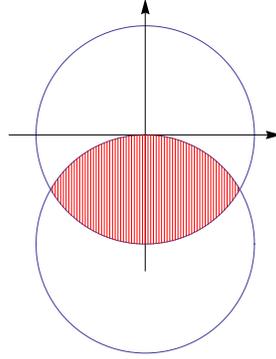
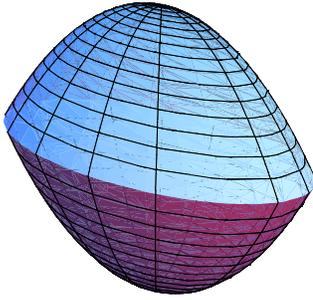
The picture shows an unidentified flying object (UFO). Although it is unidentified, we know its shape. One part of the surface

$$x^2 + y^2 + z^2 = 4$$

and the other part of the surface is

$$x^2 + y^2 + (z + 2)^2 = 4.$$

Find the surface area of the UFO.



Problem 10) (10 points)

Evaluate the following integral

$$\int_0^2 \int_1^3 \int_{z^2}^4 xz \cos(y^2) dy dx dz .$$

Problem 11) (10 points)

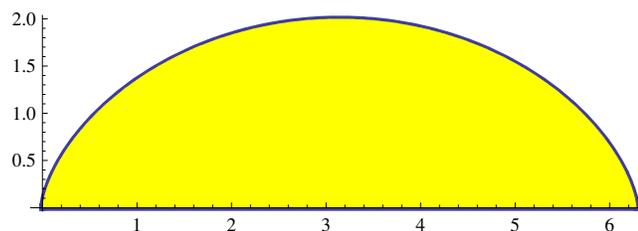
Let  $\vec{F}(x, y, z) = \langle x + yz, xye^{-xz}, e^{-xz} \rangle$ . Find

$$\iint_S \vec{F} \cdot d\vec{S} ,$$

where  $S$  is the surface  $z = 1 - x^2 - y^2, z \geq 0$  oriented so that the normal vector points upwards.

Problem 12) (10 points)

Find the area of the region on the plane enclosed by the curve  $\vec{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$  with  $0 \leq t \leq 2\pi$  and the  $x$ -axes.



Problem 13) (10 points)

Evaluate the integral

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} ,$$

where  $\vec{F}(x, y, z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$  and where  $S$  is the part of the ellipsoid  $x^2 + y^2/4 + (z + 1)^2 = 2, z > 0$  oriented so that the normal vector points upwards.

Problem 14) (10 points)

Let  $E$  be the rectangular solid  $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq 1$  and let  $S$  be the boundary of  $E$ . The surface  $S$  consists of 6 planar pieces where each is oriented so that the normal vector points outwards. Given the vector field

$$\vec{F} = \langle -x^2 - 4xy, -yz, 12z \rangle ,$$

for which parameters  $a, b$  is the flux integral

$$\int \int_S \vec{F} \cdot d\vec{S}$$

a global maximum?