Name:

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MWF 9 Dennis Tseng
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MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Print your name in the above box and **check your section**.
- Do not detach pages or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

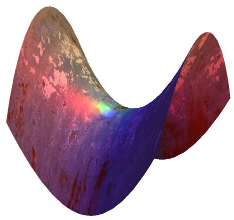
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points). No justifications are needed.

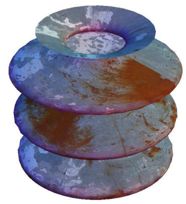
1)	T	F	If $f(x,y,z)$ is a function then the line integral of $\mathrm{curl}(\nabla f))$ around any closed circle is zero.
2)	T	F	If E is the solid half-sphere $x^2+y^2+z^2 \leq 1, z < 0$ then $\iiint_E x^4 \; dxdydz$ is positive.
3)	T	F	If the vector field $\vec{F}$ is incompressible and S and R are surfaces with the same boundary C and orientation then $\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot d\vec{S}$.
4)	T	F	$\vec{r}(u,v) = \langle \cos(u), \sin(u), 0\rangle + v\langle -\sin(u), \cos(u), 1\rangle$ parametrizes a surface in the one-sheeted hyperboloid $x^2+y^2-z^2=1$.
5)	T	F	The equation $\mathrm{div}(\mathrm{grad} f) = \|\mathrm{grad} f\|^2$ is an example of a partial differential equation for the unknown function $f(x,y,z)$.
6)	T	F	The vector $(\vec{i} \times \vec{j}) \times \vec{i}$ is the zero vector.
7)	T	F	The length of the gradient $\|\nabla f\|$ is minimal at a local minimum of $f(x,y)$ if a local minimum of f exists.
8)	T	F	The length of the gradient $\|\nabla f\|$ is maximal at a local maximum of $f(x,y)$ if a local maximum of f exists.
9)	T	F	If $\vec{r}(t)$ parametrizes the curve obtained by intersecting $y=0$ with $x^2+y^2+z^2=1$, then the bi-normal vector $\vec{B}$ is tangent to the surface.
10)	T	F	If the line integral of $\vec{F}$ along the closed loop $x^2+y^2=1, z=0$ is zero then the vector field is conservative.
11)	T	F	The vectors $\vec{v} = \langle 1,0,0\rangle$ and $\vec{w} = \langle 1,1,0\rangle$ have the property that $\vec{v} \cdot \vec{w} = \|\vec{v} \times \vec{w}\|$.
12)	T	F	The surface area of a sphere depends on the orientation of the sphere. It is positive if the normal vector points outwards and changes sign if the orientation is changed.
13)	T	F	If $f(x,y,z) = (\mathrm{div}(\vec{F})(x,y,z))$ has a maximum at $(0,0,0)$, then $\mathrm{grad}(\mathrm{div}(\vec{F}))(0,0,0) = \langle 0,0,0\,\rangle$.
14)	T	F	The curl of a conservative vector field is zero.
15)	T	F	The flux of the curl of $\vec{F}$ through a disc $x^2+y^2 \leq 1, z=0$ is always zero.
16)	T	F	If the integral $\int\int\int_G \mathrm{div}(\vec{F}(x,y,z)) \; dxdydz$ is zero for the ball $G = \{x^2+y^2+z^2 \leq 1\}$, then the divergence is zero at $(0,0,0)$.
17)	T	F	The value $\sqrt{101 \cdot 10002}$ can by linear approximation be estimated as $1000 + 5 \cdot 1 + (1/20) \cdot 2$.
18)	T	F	If $\vec{F} = \mathrm{curl}(\vec{G})$ and $\mathrm{div}(\vec{F}) = 0$ everywhere in space, then $\mathrm{div}(\vec{G}) = 0$ everywhere in space.
19)	T	F	It is possible that $\vec{v} \cdot \vec{w} > 0$ and $\vec{v} \times \vec{w} = \vec{0}$.
20)	T	F	The directional derivative $D_{\vec{v}}(f)$ is defined as $\nabla f \times \vec{v}$.

Problem 2) (10 points) No justifications are necessary.

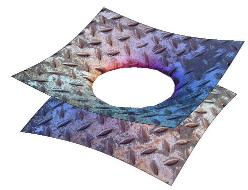
a) (3 points) The following surfaces are given either as a parametrization or implicitly in some coordinate system (Cartesian, cylindrical or spherical). Each surface matches exactly one definition.



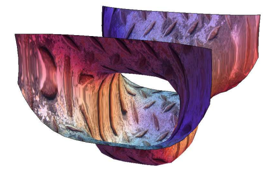
A



B



C

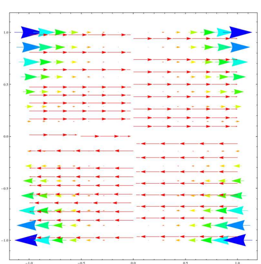


D

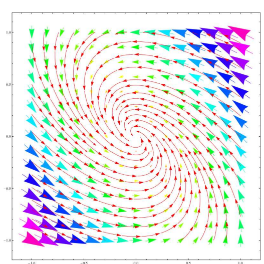
Enter A-D here	Function or parametrization
	$r = 3 + 2\sin(3z)$
	$\vec{r}(u,v) = \langle u, v, u^2 - v^2 \rangle$
	$x^4 - zy^4 + z^4 = 1$
	$r^2 - 8z^2 = 1$

b) (3 points) The pictures display flow lines of vector fields in two dimensions. Match them.

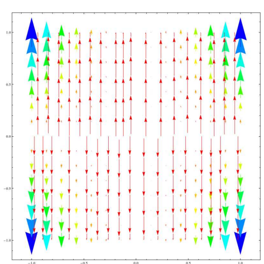
Field	Enter 1-4
$\vec{F}(x,y) = \langle 0, x^2 y \rangle$	
$\vec{F}(x,y) = \langle x^2 y, 0 \rangle$	
$\vec{F}(x,y) = \langle -y - x, x \rangle$	
$\vec{F}(x,y) = \langle -y, x \rangle$	



1



2



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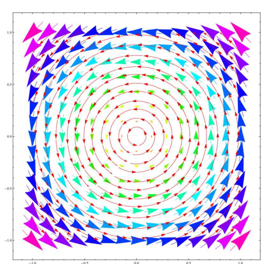


4

c) (2 points) Match the following partial differential equations with functions $u(t,x)$ which satisfy the differential equation and with formulas defining these equations.

equation	A-C	1-3
Laplace		
wave		
heat		

A	$u(t,x) = t + t^2 - x^2$
B	$u(t,x) = t + t^2 + x^2$
C	$u(t,x) = x^2 + 2t$

1	$u_t(t,x) = u_{xx}(t,x)$
2	$u_{tt}(t,x) = -u_{xx}(t,x)$
3	$u_{tt}(t,x) = u_{xx}(t,x)$

d) (2 points) Two of the six expressions are **not** independent of the parametrization. Check them.

Velocity $\vec{r}\,'(t)$		Surface area $\int\int |\vec{r}_u \times \vec{r}_v| dudv$		Line integral $\int_a^b \vec{F} \cdot d\vec{r}$	
Arc length $\int |\vec{r}\,'(t)|\, dt$		Flux integral $\int\int_R \vec{F} \cdot d\vec{S}$		Normal vector $\vec{r}_u \times \vec{r}_v$	

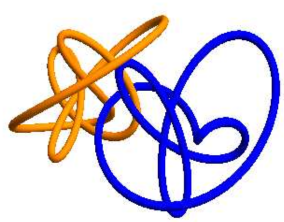
Problem 3) (10 points)

a) (4 points) The following objects are defined in three dimensional space.
Fill in either "surface", "curve", or "vector field" in each case.

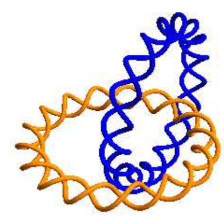
formula	surface, curve or field?
$x + y = 1$	
$x + y = 1, x - y = 5$	
$\vec{F}(x, y, z) = \langle x, x + y, x - y \rangle$	
$\vec{r}(x, y) = \langle x, y, x - y \rangle$	
$\vec{r}(x) = \langle x, x, x^2 - x \rangle$	

b) (2 points) Two closed curves $\vec{r}_1(t), \vec{r}_2(t)$ form a **link**. In our case, the curve $\vec{r}_2(t)$ is a copy of the other moved and turned around. Match three of them. Links and knots are relevant in biology: DNA strands can forms links or knots which need disentanglement.

$\vec{r}_1(t)$	Enter A,B,C,D
$\langle 7\cos(t), 7\sin(t), 7\cos(t) \rangle$	
$\langle (7 + \cos(17t))\cos(t), (7 + \cos(17t))\sin(t), \sin(17t) \rangle$	
$\langle \cos(2t) + \sin(4t), \cos(4t) + \cos(3t), \cos(2t) + \sin(3t) \rangle$	



A



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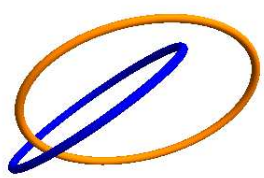

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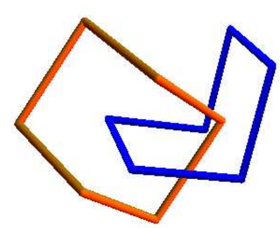
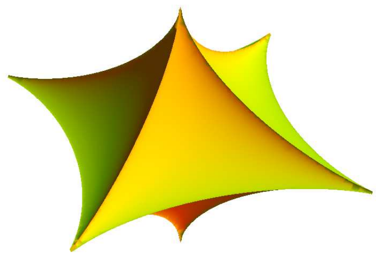


D

c) (4 points) Which of the following expressions are defined if $\vec{F}(x, y, z)$ is a vector field and $f(x, y, z)$ a scalar field in space. Is the result a scalar or vector field?

Formula	Defined	Not defined	Scalar	Vector
$\mathrm{curl}(\mathrm{grad}(\mathrm{div}(\vec{\mathrm{F}})))$				
$\mathrm{curl}(\mathrm{div}(\mathrm{grad}(\mathrm{f})))$				
$\mathrm{grad}(\mathrm{div}(\mathrm{curl}(\vec{\mathrm{F}})))$				
$\mathrm{grad}(\mathrm{curl}(\mathrm{div}(\vec{\mathrm{F}})))$				
$\mathrm{div}(\mathrm{curl}(\mathrm{grad}(\mathrm{f})))$				
$\mathrm{div}(\mathrm{grad}(\mathrm{curl}(\mathrm{f})))$				

Problem 4) (10 points)



a) (5 points) Find the tangent plane to the surface S given by

$$3x^{2/3} + 3y^{2/3} + 6z^{2/3} = 12$$

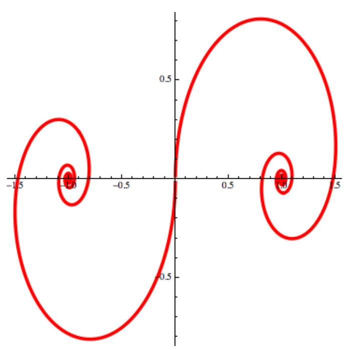
at the point $(1, 1, 1)$.

b) (5 points) When S is intersected with the plane $y = 1$, we get the curve

$$3x^{2/3} + 6z^{2/3} = 9 \ .$$

Find the tangent line of the form $ax + bz = d$ for the tangent line at $(x, z) = (1, 1)$.

Problem 5) (10 points)



Find the line integral for the vector field

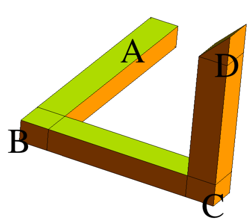
$$\vec{F}(x, y) = \langle x^6 + y + 3x^2y^3, y^7 + x + 3x^3y^2 \rangle$$

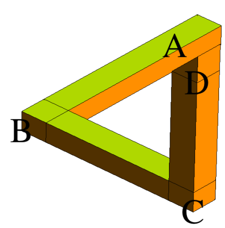
along the **ornamental curve**

$$\vec{r}(t) = \langle \frac{t}{|t|}(1 - \frac{1}{1+t^2}\cos(4t)), \frac{1}{1+t^2}\sin(4t) \rangle$$

from $t = -\infty$ to $t = \infty$. This curve connects the point $(-1, 0)$ with $(1, 0)$ along an **infinite epic journey**.

Problem 6) (10 points)





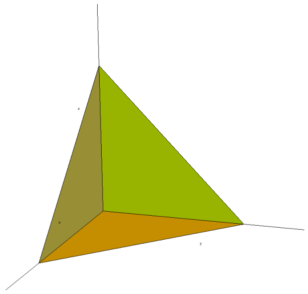
The **Penrose tribar** is a path in space connecting the points $A = (1, 0, 0), B = (0, 0, 0), C = (0, 0, 1), D = (0, 1, 1)$. While the distance between A and D is positive, we see an impossible triangle when the line of sight goes through A and D.

a) (2 points) Find the distance between the points A and D.

b) (4 points) Parametrize the line connecting A and D.

c) (4 points) Find the distance between the lines AB and CD.

Problem 7) (10 points)



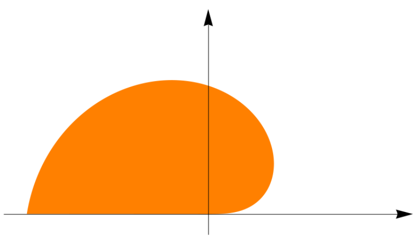
You invent a **3D printing process** in which materials of variable density can be printed. To try this out, we take a tetrahedral region E:

$$x + y + z \leq 1; x \geq 0, y \geq 0, z \geq 0$$

which has the density $f(x, y, z) = 24x$. Find the total mass

$$\int\int\int_E f(x, y, z) \, dxdydz \, .$$

Problem 8) (10 points)



When we integrate the function $f(x, y) = 2y/(\sqrt{x^2 + y^2} \arctan(y/x))$ over the **snail region** $r^2 \leq \theta \leq \pi$, we are led in polar coordinates to the integral

$$\int_0^{\sqrt{\pi}} \int_{r^2}^{\pi} \frac{2r \sin(\theta)}{\theta} \, d\theta \, dr$$

Evaluate this integral.

Problem 9) (10 points)

Old Mc Donald wants to build a farm on a location, where the ground is as even as possible. Let $g(x, y) = y^2 + xy + x$ be the height of the ground. Find the point (x, y), where the **steepness**

$$f(x, y) = |\nabla g|^2$$

is minimal. Classify all critical points of f.

Problem 10) (10 points)

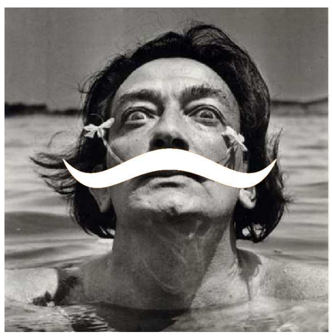
We build a **bike** which has as a frame a triangle of base length x and height y and a wheel which has radius y. Using Lagrange, find the bike which has maximal

$$f(x,y) = xy + 4\pi y^2$$

(which is twice the area) under the constraint

$$g(x,y) = x + 10\pi y = 3 \ .$$

Problem 11) (10 points)



Compute the area of the **moustache region** which is enclosed by the curve

$$\vec{r}(t) = \langle 5\cos(t), \sin(t) + \cos(4t) \rangle$$

with $0 \leq t \leq 2\pi$.

Hint. You can use without justification that integrating an odd 2π periodic function from 0 to 2π is zero.

Problem 12) (10 points)

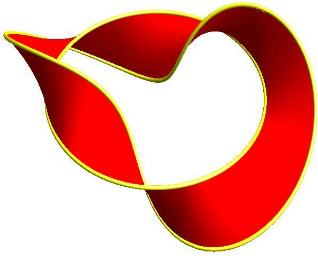
We enjoy the pre-holiday season in a local Harvard square coffee shop, where **coffee aroma** diffuses in the air. Find the flux of the air velocity field

$$\vec{F}(x,y,z) = \langle y^2, x^2, z^2 \rangle$$

leaving a coffee box

$$E : x^2 + y^2 \leq 1, x^2 + y^2 + z^2 \leq 4 \ .$$

Problem 13) (10 points)



Find the line integral of

$$\vec{F}(x,y,z) = \langle -y, x, e^{\sin z} \rangle$$

along the positively oriented boundary of the **ribbon** $\vec{r}(u,v)$ parametrized on $0 \leq u \leq 4\pi$ and $0 \leq v \leq 1/2$ with

$$\vec{r}(u,v) = \langle (1+v\cos(2u))\cos(u), (1+v\cos(2u))\sin(u), v\sin(2u) \rangle$$

for which a **good fairy** gives you the normal vector

$$\begin{aligned} \vec{r}_u \times \vec{r}_v \quad = \quad & \langle -\sin(u)(v\cos(4u) + 2(v+1)\cos(2u) - 3v + 2)/2 \ , \\ & \cos(u)(v\cos(4u) - 2(v-1)\cos(2u) - 3v - 2)/2 \ , \\ & -\cos(2u)(v\cos(2u) + 1) \rangle \ . \end{aligned}$$

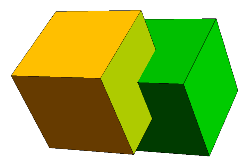
Problem 14) (10 points)

A computer can determine the volume of a solid enclosed by a triangulated surface by computing the fluxe of the vector field $\vec{F} = \langle 0, 0, z \rangle$ through each triangle and adding them all up. Lets go backwards and compute the flux of this vector field $\vec{F} = \langle 0, 0, z \rangle$ through the surface S which bounds a solid called "abstract cow" (this is avant-garde **"neo-cubism"** style)

$$\{0 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq 2\} \ \cup$$

$$\{1 \leq x \leq 3, 1 \leq y \leq 3, 1 \leq z \leq 3\} \ ,$$

where $\cup$ is the union and the surface is oriented outwards.



Ceci n'est pas une pipe

Ceci c'est une vache