

Name:

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- 1) T F For any continuous function $f(x, y)$, we have $\int_0^1 \int_1^2 f(x, y) dx dy = \int_1^2 \int_0^1 f(x, y) dx dy$.

Solution:

This is not Fubini

- 2) T F If \vec{u} is a unit vector tangent to $f(x, y) = 1$ at $(0, 0)$ and $f(0, 0) = 1$, then $D_{\vec{u}}f(0, 0)$ is zero.
- 3) T F Assume f is zero on $x = y$ and $x = -y$, then $(0, 0)$ is a critical point of f .
- 4) T F If $(0, 0)$ is the only local minimum of a function f and f has no local maxima, then $(0, 0)$ is a global minimum.
- 5) T F If $(0, 0)$ is a critical point for f , and $f_{yy}(0, 0) < 0$ then $(0, 0)$ is not a local minimum.
- 6) T F If $f(x, y)$ and $g(x, y)$ have the same non-constant linearization $L(x, y)$ at $(0, 0)$ and $f(0, 0) = g(0, 0) = 0$, then the level sets $f = 0$ and $g = 0$ have the same tangent line at $(0, 0)$.
- 7) T F There are saddle points with positive discriminant $D > 0$.

Solution:

No. By definition not

- 8) T F If R is the unit disc, then $\int_R x^2 - y^2 dx dy$ is zero.

Solution:

Yes. This can be seen as the definition of area.

- 9) T F There is a nonzero function $f(x, y)$ for which the linearization $L(x, y)$ is equal to $2f(x, y)$.

Solution:

The condition means that the function is linear. For a linear function, the linearization is itself.

- 10) T F The directional derivative at a local minimum $(0, 0)$ is positive in every direction.

Solution:

It is zero

- 11) T F If $\vec{r}(t)$ is a curve on the surface $g(x, y, z) = 1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}'(t) = 0$.

Solution:

Use the chain rule and the fact that $g(\vec{r}(t))$ is constant so that $d/dtg(\vec{r}(t))$ is zero.

- 12) T F If $|\nabla f(0, 0)| = 2$, there is a direction in which the directional derivative at $(0, 0)$ is 2.

Solution:

Yes, we have seen this computation

- 13) T F If $D > 0$ at $(0, 0)$ and $\nabla f(0, 0) = 0$ and $f_{xx}(0, 0) < 0$ then $f_{yy}(0, 0) < 0$.

Solution:

We have seen this from $D = f_{xx}f_{yy} - f_{xy}^2$.

- 14) T F $\int_0^1 \int_0^x f(x, y) dy dx = \int_0^1 \int_y^1 f(x, y) dx dy$.

Solution:

This is almost a correct correct switch but the $dx dy$ have also to be switched.

- 15) T F The surface area of the sphere of radius L is $\int_0^\pi L^2 \sin(\phi) d\phi$.

Solution:

The answer is 4π .

- 16) T F If $f(x, y) = g(x)$ is a function of x only, then $D = 0$ at every critical point.

Solution:

Indeed, then $f_{xy}, f_{xx}f_{yy}$ are both zero.

- 17) T F The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the surface $z = f(x, y)$.

Solution:

Nonsense

- 18) T F If $|\nabla f(0, 0)| = 2$, then there is a unit vector \vec{v} such that $D_{\vec{v}}f(0, 0) = 1$.

Solution:

Intermediate value theorem.

- 19) T F The gradient of the function $f(x, y) = \int_x^y \sin(t) dt$ is $\langle -\sin(x), \sin(y) \rangle$.

Solution:

Yes, this is the derivative by the fundamental theorem.

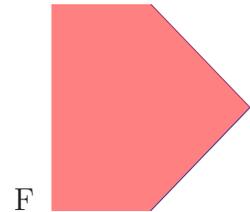
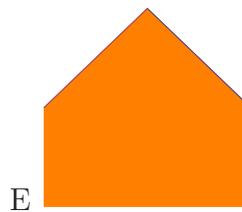
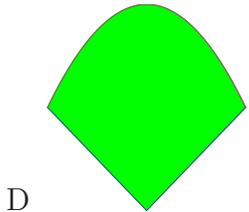
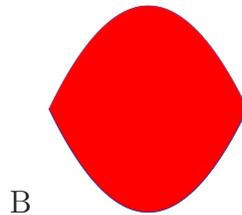
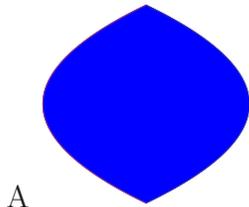
- 20) T F Assume $f(x, y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt}f(\vec{r}(t)) \geq 0$.

Solution:

FTTFTTFTFFTTTTFTFTTT

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region A–F.



Enter A-F	Integral
	$\int_{-1}^1 \int_{-1}^{2- y } f(x, y) \, dx dy$
	$\int_{-1}^1 \int_{y^2}^{2- y } f(x, y) \, dx dy$
	$\int_{-1}^1 \int_{x^2}^{2-x^2} f(x, y) \, dy dx$
	$\int_{-1}^1 \int_{ x }^{2-x^2} f(x, y) \, dy dx$
	$\int_{-1}^1 \int_{y^2}^{2-y^2} f(x, y) \, dx dy$
	$\int_{-1}^1 \int_{-1}^{2- x } f(x, y) \, dy dx$

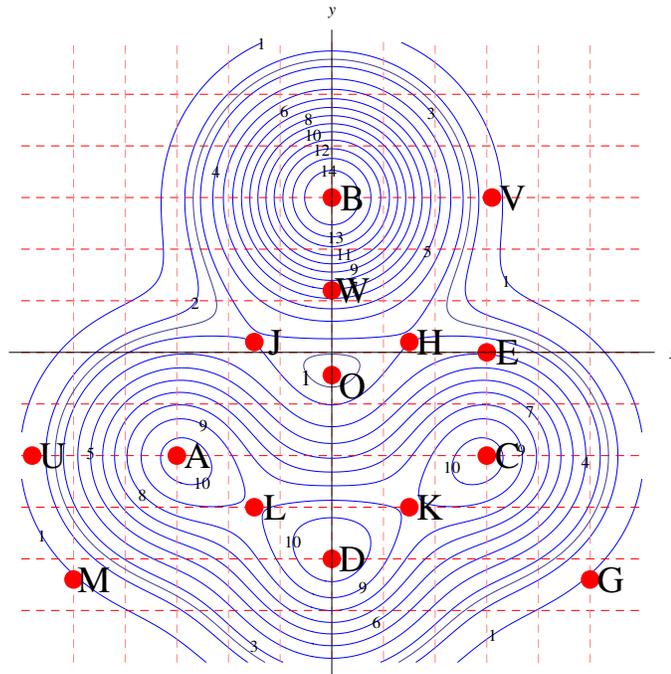
b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation Number	PDE
1	$g_x - g_y = 0$
2	$g_{xx} - g_{yy} = 0$
3	$g_x - g_{yy} = 0$
4	$g_{xx} + g_{yy} = 0$

Solution:
 FCBDAE
 4,2,1,3

Problem 3) (10 points)



a) (6 points) Enter one label into each of the boxes.

At which point is the length of the gradient maximal?

At which point is the global maximum?

At which point is $f_x > 0, f_y = 0$?

At which point is $D_{\langle 1,1 \rangle/\sqrt{2}}f = 0, D_{\langle 1,-1 \rangle/\sqrt{2}}f < 0$?

At which point is f maximal under the constraint $g(x, y) = y = 0$?

At which point does f have a local minimum?

Solution:

b) (4 points) Note that the zero vector is considered both parallel and perpendicular to any other vector.

	parallel	perp	
The gradient ∇f is always			to the surface $f = c$.
For a Lagrange minimum, ∇g is			to ∇f .
If $(0, 0)$ is a min. of f then $\nabla f(0, 0)$ is			to $\langle 1, 0 \rangle$.
If $(0, 0)$ is max. of f and $g = z - f(x, y)$ then ∇g is			to $\langle 0, 0, 1 \rangle$.

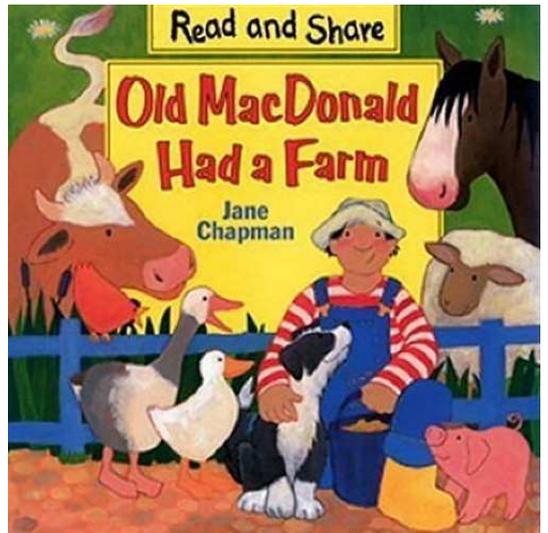
Solution:

WBUGEO

	parallel	perp	
The gradient ∇f is always		*	to the surface $f = c$.
For a Lagrange minimum, ∇g is	*		to ∇f .
If $(0, 0)$ is a min. of f then $\nabla f(0, 0)$ is	*	*	to $\langle 1, 0 \rangle$.
If $(0, 0)$ is max. of f and $g = z - f(x, y)$ then ∇g is	*		to $\langle 0, 0, 1 \rangle$.

Problem 4) (10 points)

A farm costs $f(x, y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all x, y . Estimate $f(12, 19)$.



"Old MacDonald had a million dollar farm, E-I-E-I-O, and on that farm he had $x = 10$ cows, E-I-E-I-O, and on that farm he had $y = 20$ ducks, E-I-E-I-O, with $f_x = 2x$ here and $f_y = y^2$ there, and here two cows more, and there a duck less, how much does the farm cost now, E-I-E-I-O?"

Solution:

$f(10, 20) = 1000000$. The linearization is $L(12, 19) = f(10, 20) + 2(10)(12 - 10) + (20)^2(19 - 20) = 1000000 + 2(20) - 400 = 999640$.

Problem 5) (10 points)

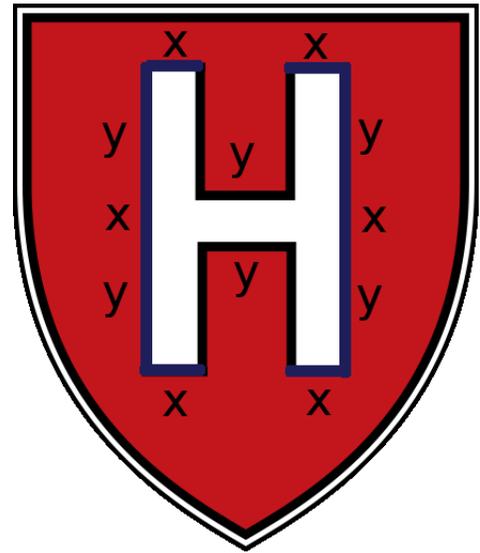
Find the Harvard H which has maximal area

$$f(x, y) = 5xy + 2x^2$$

with fixed exposed perimeter

$$6x + 4y = 88 .$$

Find the maximum using Lagrange.



Solution:

The Lagrange equations are

$$5y + 4x = 6\lambda$$

$$5x = 4\lambda$$

$$6x + 4y = 88$$

Eliminating λ gives $y = (7/10)x$. $x = 10, y = 7, f(10, 7) = 550$.

Problem 6) (10 points)

a) (7 points) A minigolf on the cape has a hole at a local minimum of the function

$$f(x, y) = 3x^2 + 2x^3 + 2y^5 - 5y^2 .$$

Find all the critical points and classify them.

b) (3 points) A golfer hits tangent to the level curve $f(x, y) = 2$ through $(1, 1)$. Find this line.

About minigolf: the first standardized minigolf course appeared in 1916 in North Carolina. The world record on a round of minigolf is 18 strokes on 18 holes on eternite. No perfect round on concrete has been scored. The highest prizes reach 5000 dollars only so that nobody is known to make a living by competing in minigolf.



Solution:

	-1	0	60	-6	max	1
a)	-1	1	-180	-6	saddle	-2
	0	0	-60	6	saddle	0
	0	1	180	6	min	-3

b) We know the gradient at the point. $x = 1$.

Problem 7) (10 points)

A circular track near Salem is a circle of radius 500 which is centered at the origin $(0, 0)$. A go-kart goes counter-clockwise around the track $\vec{r}(t)$. The cheering intensity is given by a function $f(x, y)$. The go-kart passes the point $(300, 400)$ at time $t = 0$ with velocity $\langle -4, 3 \rangle$. We know that $f_x(300, 400) = 2$ and $f_y(300, 400) = 10$. Find the rate of change

$$\frac{d}{dt}f(\vec{r}(t))$$

at $t = 0$.



Solution:

Use the chain rule:

$$\frac{d}{dt}f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

At $t = 0$, we have $\langle 2, 10 \rangle \cdot \langle -4, 3 \rangle = -8 + 30 = 22$.

Problem 8) (10 points)

a) (6 points) Find the integral

$$\int_0^1 \int_y^{y^{1/5}} \frac{e^x + x^7}{x - x^5} dx dy .$$

b) (4 points) Integrate

$$\int_{-1}^0 \int_0^{\sqrt{1-y^2}} \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} dx dy .$$

Solution:

a) Change the order of integration $e - 7/8$

b) Use polar coordinates $\pi/2(e - 1)$. Note that the region is in the fourth quadrant so that we integrate from $-\pi/2$ to 0.

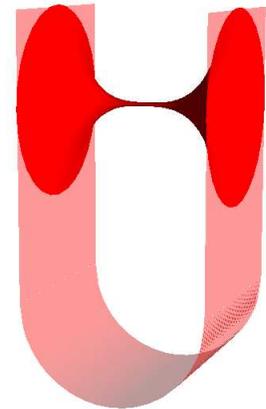
Problem 9) (10 points)

Find the surface area of the "wormhole"

$$\vec{r}(u, v) = \langle 3v^3, v^9 \cos(u), v^9 \sin(u) \rangle,$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Einstein-Rosen bridges are hypothetical topological constructions which would allow shortcuts through space-time. Tunnels connecting different parts of the universe appear frequently in science fiction.



Solution:

Note that $\int \int |\vec{r}_u \times \vec{r}_v| \, dudv$ has a nonnegative integrand. In our case it is $9v^{11}\sqrt{1+v^{12}}$. When we integrate this from 0 to 2π we get $18\pi v^{11}\sqrt{1+v^{12}}$. But note that we have to take either square root so that the integrand is nonnegative or just take the absolute value. In any case, we can just take twice the integral from 0 to 1. The answer is not zero but $2\pi(\sqrt{8} - 1)$.

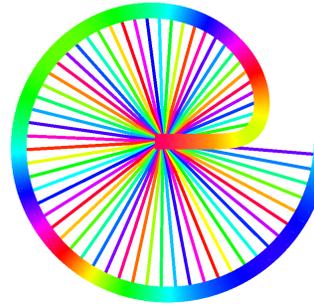
Problem 10) (10 points)

a) (5 points) We become typographer and design new mathematically defined **type-face** of the alphabet. The new letter "e" in this "21a" design is given by a polar region $r(t) \leq t^{1/7}$, with $0 \leq t \leq 2\pi$. Find the area of this region.

b) (5 points) Integrate

$$\int_0^1 \int_0^{\arccos(y)} \frac{1}{\cos(x)} dx dy .$$

Remark: Computer scientist **Donald Knuth** once wrote an entire article about "The Letter S".



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The Letter S
Donald E. Knuth

SEVERAL YEARS AGO when I began to look at the problem of designing suitable alphabets for use with modern printing equipment, I found that 22 of the letters were comparatively easy to deal with. The other letter was 'S'. For three days and nights I had a terrible time trying to understand how a proper 'S' could really be defined. The solution I finally came up with turned out to involve some interesting mathematics, and I believe that students of calculus and analysis generally may enjoy looking into the question as I did. The purpose of this paper is to explain what I now consider to be the 'right' mathematical definition of a printed 'S', and also to give an example of the METAFONT language I have recently been developing. [A complete description of METAFONT, which is a non-proprietary system and language intended to aid in the design of letter shapes, appears in [1], part 2].

Before getting into a technical discussion, I should probably mention why I started worrying about such things in the first place. The central issue is that today's printing technology is essentially based on discrete mathematics and computer science, not on properties of continua or of smooth type. The task of making a plate for a printed page is now essentially that of constructing a square matrix of 0's and 1's, where the 0's specify white space and the 1's specify ink. I would like to avoid a case of this kind, but I had been trying with the old hand-set technology, and when I realized that this problem could be solved by using appropriate techniques of discrete mathematics and computer science, I couldn't resist trying to find my own solution.

Reference [2] explains more of the background of my work, and I also discuss the early history of mathematical approaches to type design. In particular, it illustrates how several people proposed to construct 'S's

...che quella tavola può far la sua parte, lo considero del quadro, lungo di là inferiore, fino ad uguale spazio verso il basso, per essere però a. perché non possa dare il suo la inferiore parte, ed è, per far fare altri linee, che lungo di là linea del tipo di parte destra, non a, non

perché a. da la inferiore inferiore del quadro. L'altra parte lungo da quella del quadro da parte sinistra, però a. discesa in modo verso una data parte che segue sopra la medesima linea. Per una altra larghezza da questa parte, una parte dove il presente finire, l'altra parte lungo da la linea del quadro da parte sinistra, verso la destra del quadro, come che da lontano da la inferiore linea del quadro parte a. Per da questa ultima parte, si manda verso la linea linea a compingere con la inferiore parte lungo da la linea da parte sinistra del quadro parte a, verso verso: o' nel libro la lettera S, come appertamente a vede.

Fig. 2. Francesco Torricelli's method of "squaring the π " in 1577. (This is page 54 of [3], reproduced by kind permission of Ottavio Bazzani in Venice, Italy.)

Solution:

- a) We have $\int_0^{2\pi} t^{2/7} / 2 dt = (7/18)(2\pi)^{9/7}$.
- b) Change the order of integration. We have

$$\int_0^{\pi/2} \int_0^y \frac{1}{\cos(x)} dy dx = \int_0^{\pi/2} 1 dy = \pi/2 .$$